# Math 322 Lecture 10

Well-posedness for Laplace's equation with Dirichlet boundary conditions

Idea: A problem is well posed if the solution varies a small amount for small changes in the boundary or initial data

Consider $\nabla^2 u = 0$ inside $R$
$u = f(\underline{x})$ on the boundary of $R$

Now make a small change in boundary data
$\Longrightarrow$ there is a different solution $v$ satisfying

$\nabla^2 v = 0$ inside $R$
$v = g(\underline{x})$ on the boundary of $R$

with $f(\underline{x}) - g(\underline{x})$ small at all points on boundary

Now consider $w = u - v$:

$w$ satisfies $\nabla^2 w = \nabla^2(u-v) = \nabla^2 u - \nabla^2 v = 0 - 0 = 0$
where linearity has been used

with $w(\underline{x}) = u(\underline{x}) - v(\underline{x}) = f(\underline{x}) - g(\underline{x}) \equiv h(\underline{x})$
specified on the boundary

At all points in $R$, the max/min principle gives

$$h_{min} \le w \le h_{max}$$

$$\min(f-g) \le w \le \max(f-g)$$

so we can make $w$ in the interior as small as we want

Explicit Example:

$$\nabla^2 u(r,\theta) = 0 \qquad 0 \le r < a, \ -\pi < \theta \le \pi$$
$$u(a,\theta) = f(\theta)$$

$$f(\theta) = \cos 3\theta \qquad g(\theta) = \cos 3\theta - \varepsilon \cos N\theta$$

$$u(r,\theta) = \left(\frac{r}{a}\right)^3 \cos 3\theta \qquad v(r,\theta) = \left(\frac{r}{a}\right)^3 \cos 3\theta$$
$$- \varepsilon \left(\frac{r}{a}\right)^N \cos N\theta, \quad \varepsilon > 0$$

$$u(r,\theta) - v(r,\theta) = \varepsilon \left(\frac{r}{a}\right)^N \cos N\theta$$

$$-\varepsilon \le u(r,\theta) - v(r,\theta) \le \varepsilon$$

since $\left(\frac{r}{a}\right)^N \le 1 \qquad -1 \le \cos N\theta \le 1$

**Uniqueness** $\nabla^2 u = 0$ Dirichlet b.c.s

Maximum Principle again

Suppose that there are 2 solutions $u$ and $v$ satisfying

$$\nabla^2 u = 0 \text{ in } R, \quad u = f(\underline{x}) \text{ on the boundary of } R$$

$$\nabla^2 v = 0 \text{ in } R, \quad v = f(\underline{x}) \text{ on the boundary}$$

Max Principle tells us

$$0 \le w = u - v \le 0 \implies u = v$$

at all interior points

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Laplace's Equation with Dirichlet boundary conditions

* well-posed
* unique soln.

$\boxed{\text{Solvability condition for Flux boundary condition:}}$

$-K\,\underline{\nabla} u(\underline{x}, t) \circ \hat{n}$ specified on boundary

$\nabla^2 u = 0$ in interior

Integrate over the domain (2D)

$$\iint \nabla^2 u \, dx\,dy = \iint \underline{\nabla} \circ (\underline{\nabla} u)\, dx\,dy = 0$$

$$= \oint \underline{\nabla} u \circ \hat{n}\, dS$$

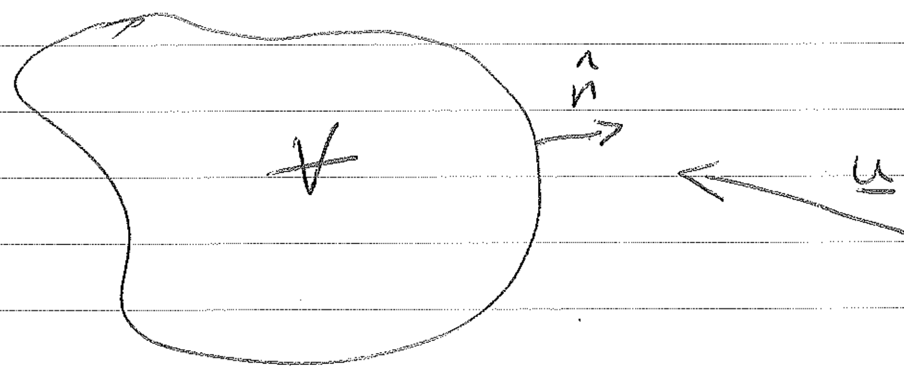
$\Rightarrow$ net heat flux through boundary must $= 0$

otherwise no solution

Laplace's Eqn. outside the circular cylinder and what's this have to do with Flow.

[ we said our heat eqn. is for heat conduction in a solid ..., now mention fluid flow around an 'airfoil ]

Lets say we wanted to consider Conservation of Energy in an arbitrary volume of fluid




Now there is a fluid velocity that can carry heat across the boundary

At the macroscopic level, there is a flux across the boundary

$$\oint_A \underline{u}(x,t)\, e(\underline{x},t)\cdot(-\hat{n})\, dA$$

Check units: $\left[\frac{\ell}{t}\right]\left[\frac{m\ell^2}{t^2}\frac{1}{\ell^3}\right]\left[\ell^2\right] = \left[\frac{m\ell^2}{t^3}\right]$

For fluid flow we need conservation of mass, momentum, energy
<u>and</u> a thermodynamic relation

$\Rightarrow$ 6 equations total for

6 scalar quantities: $T(x,t)$, $p(\underline{x},t)$, $\rho(\underline{x},t)$

and 3 components of velocity $\underline{u}(\underline{x},t)$

Consider Conservation of mass {exercise 2.5.17}

$$\frac{d}{dt}\int_{V} \rho \, dV = \oint_{A} \underline{u}\,\rho \cdot (-\hat{n})\, dA$$

$$= -\int_{V} \underline{\nabla}\cdot(\rho\,\underline{u})\, dV$$

Now shrink to a point $\Rightarrow$

$$\frac{\partial}{\partial t}\rho(\underline{x},t) + \underline{\nabla}\cdot\left(\rho(\underline{x},t)\,\underline{u}(\underline{x},t)\right) = 0$$

If $\rho$ = constant $\Rightarrow$

$$\underline{\nabla} \cdot \underline{u} = 0$$

Now repeat for momentum, energy and again assuming constant density $\Rightarrow$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\frac{1}{\rho_0} \underline{\nabla} p + g$$

$$\frac{\partial T}{\partial t} + (\underline{u} \cdot \underline{\nabla}) T = \ldots$$

Complicated coupled, nonlinear PDEs.

Lets make the very strange and very special assumption that the flow satisfies a condition

$$\underline{\nabla} \times \underline{u} = 0$$

the flow is "irrotational"

What could this mean?