Math 322 Lecture 12

Goal: To understand when

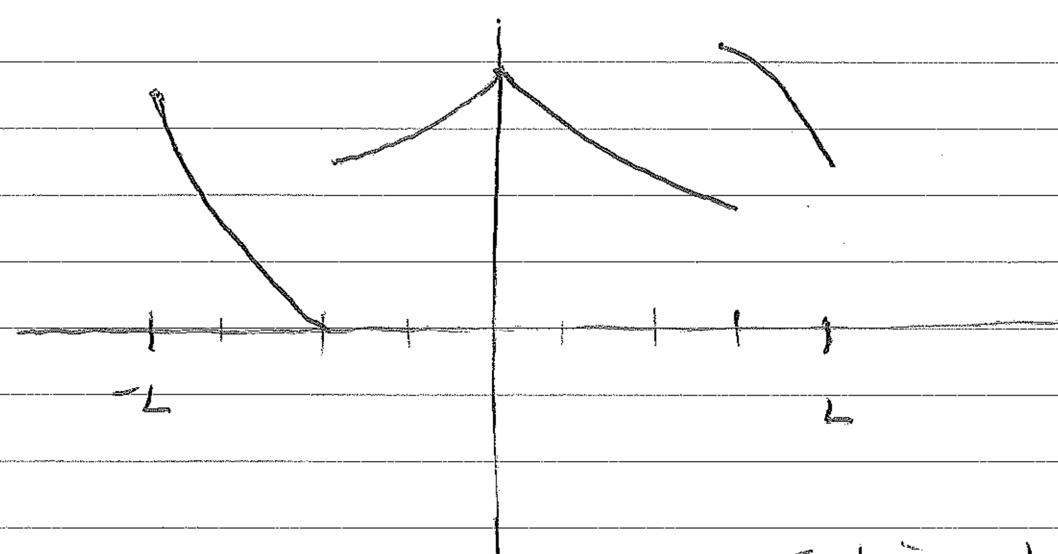
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$ "makes sense"

① Does the infinite series converge?
② Does it converge to $f(x)$

Definition: A function is piecewise smooth on $-L \le x \le L$ if

* $[-L, L]$ can be divided into pieces

* In each piece, $f(x)$ and $\frac{df(x)}{dx}$ are continuous



function values and derivatives must be finite

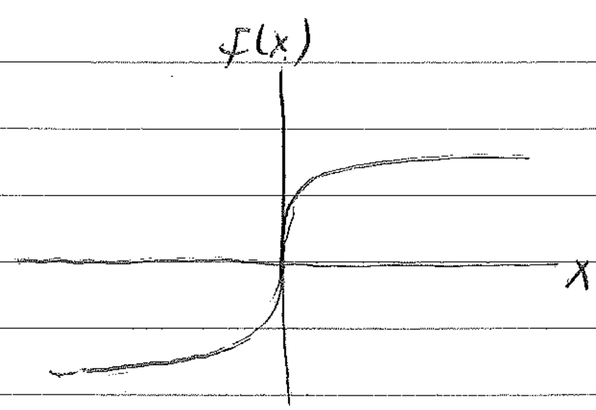
There is a jump discontinuity in $f(x)$ at $x_0 = -\frac{L}{2}$ and $x_0 = \frac{3L}{4}$

There is a jump discontinuity in $\frac{df(x)}{dx}$ at $x_0 = 0$

There are 4 intervals: $(-L, -\frac{L}{2})$ $(-L/2, 0)$ $(0, 3L/4)$ $(\frac{3L}{4}, L)$

Inside each open interval, $f(x)$ and $\frac{f(x)}{dx}$ are continuous $\Rightarrow$ the function is piecewise smooth in $-L \le x \le L$

$f(x) = x^{1/3}$ is not piecewise smooth on $-L \le x \le L$ because its derivative is not finite at $x_0 = 0$.



finite limits of $f(x)$ and $f'(x)$ are required!

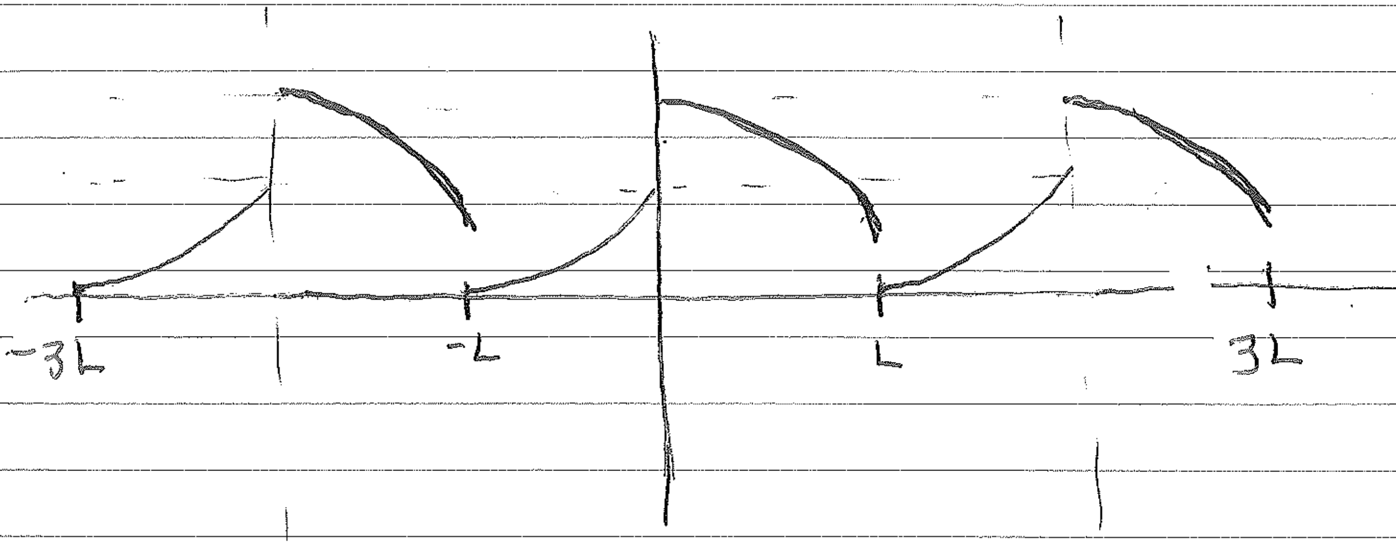
Not allowed!

In chapter 3 we always assume that $f(x)$ is piecewise smooth in $[-L, L]$!

If not stated explicitly, it is implicit ...

What is a periodic extension of $f(x)$?
(outside $-L \le x \le L$)

Repeat the shape of $f(x)$, with period $2L$



Convergence Thm If $f(x)$ is piecewise smooth on $-L \le x \le L$, then the series corresponding to

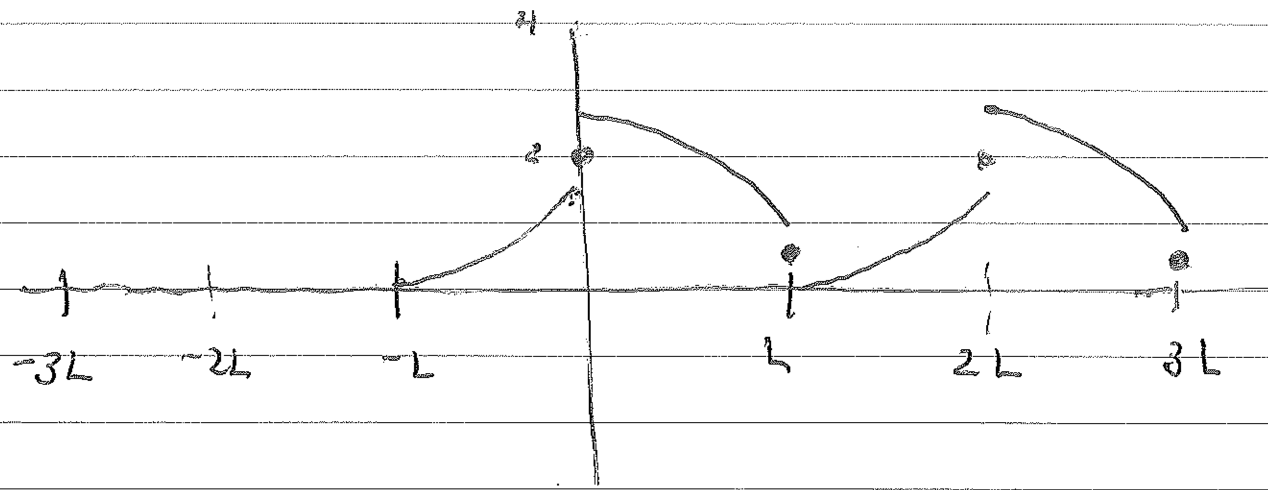
$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

converges to

① the periodic extension of $f(x)$, wherever the periodic extension is continuous

② $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ wherever the periodic extension has a jump discontinuity

In the example plot above



Jump discontinuities at $x = nL$

at $x_0 = 0$: $\dfrac{f(0^-) + f(0^+)}{2} = 2$

etc.

at $x_0 = L$: $\dfrac{f(L^-) + f(L^-)}{2} = \frac{1}{2}$

Some straightforward facts:

If $f(x)$ is even, only the a_n's will be nonzero (therefore we can start with a cosine series)

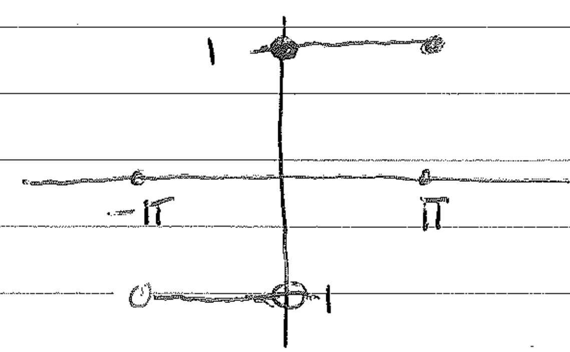
If $f(x)$ is odd, only the b_n's will be nonzero (so start with a sine series)

What is not so obvious: the behavior of the a_n's, b_n's where there is (is not) a jump discontinuity

Square Wave Example:

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \le x \le \pi \end{cases}$$

$f(x) = 1$ at $x=0$; Thm tells us that the series converges to $\frac{f(0^-)+f(0^+)}{2} = 0$.



The function is odd, so need only the b_n's

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi} = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi^+}^{0^-} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx$$

$$= \frac{1}{\pi}\left.\frac{\cos nx}{n}\right|_{-\pi^+}^{0^-} + \left.\frac{-\cos nx}{\pi n}\right|_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{n} - \frac{\cos(-n\pi)}{n}\right) - \frac{1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{1}{n}\right)$$

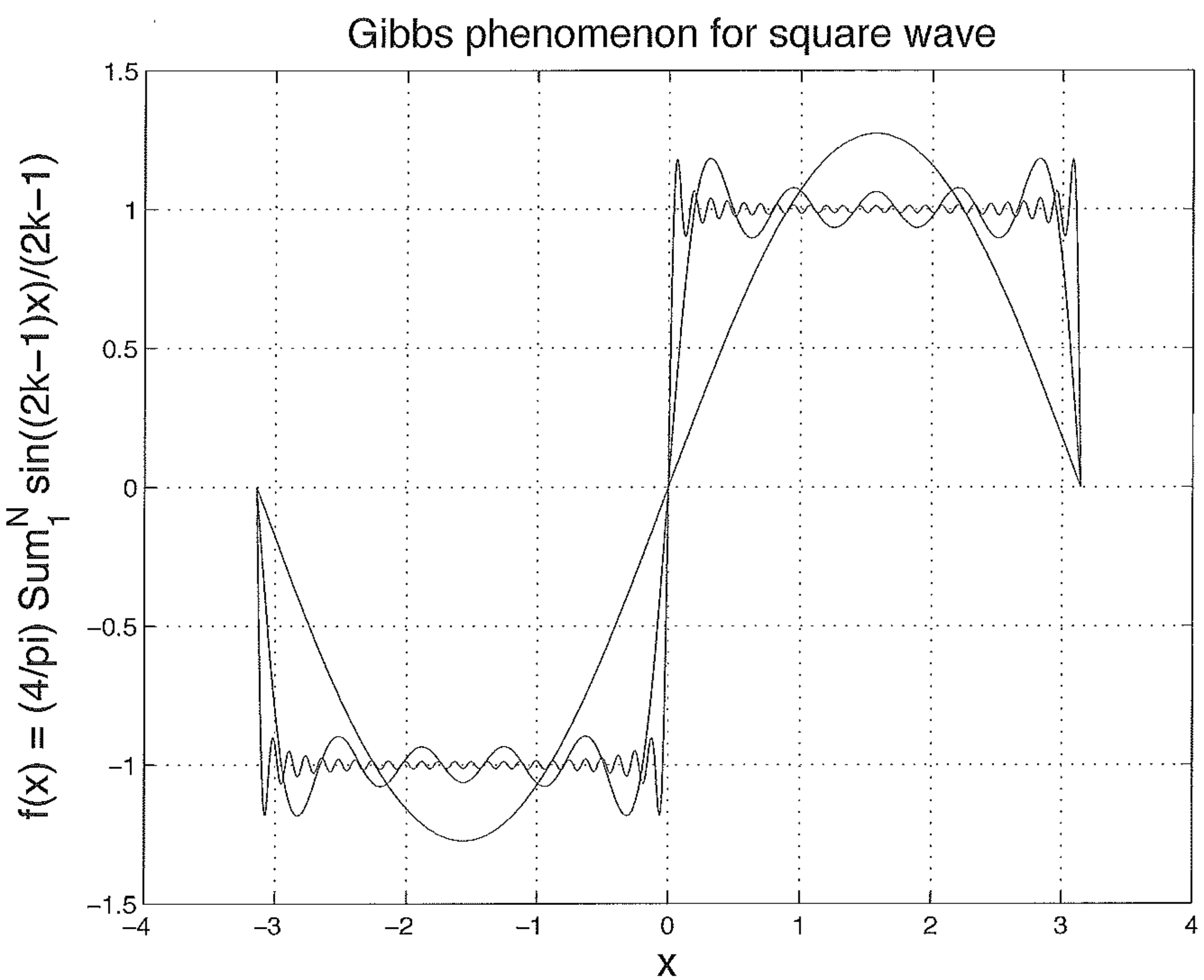
$$= \frac{1}{\pi}\left[\frac{1}{n} + \frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n}\right] = \frac{2}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n}\right]$$

$$= \begin{cases} \frac{2}{\pi}\cdot 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd} \end{cases}$$

$$f(x) \sim \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin nx}{n} = \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)}{(2k-1)}$$

$\boxed{\text{See picture of the partial sums}}$

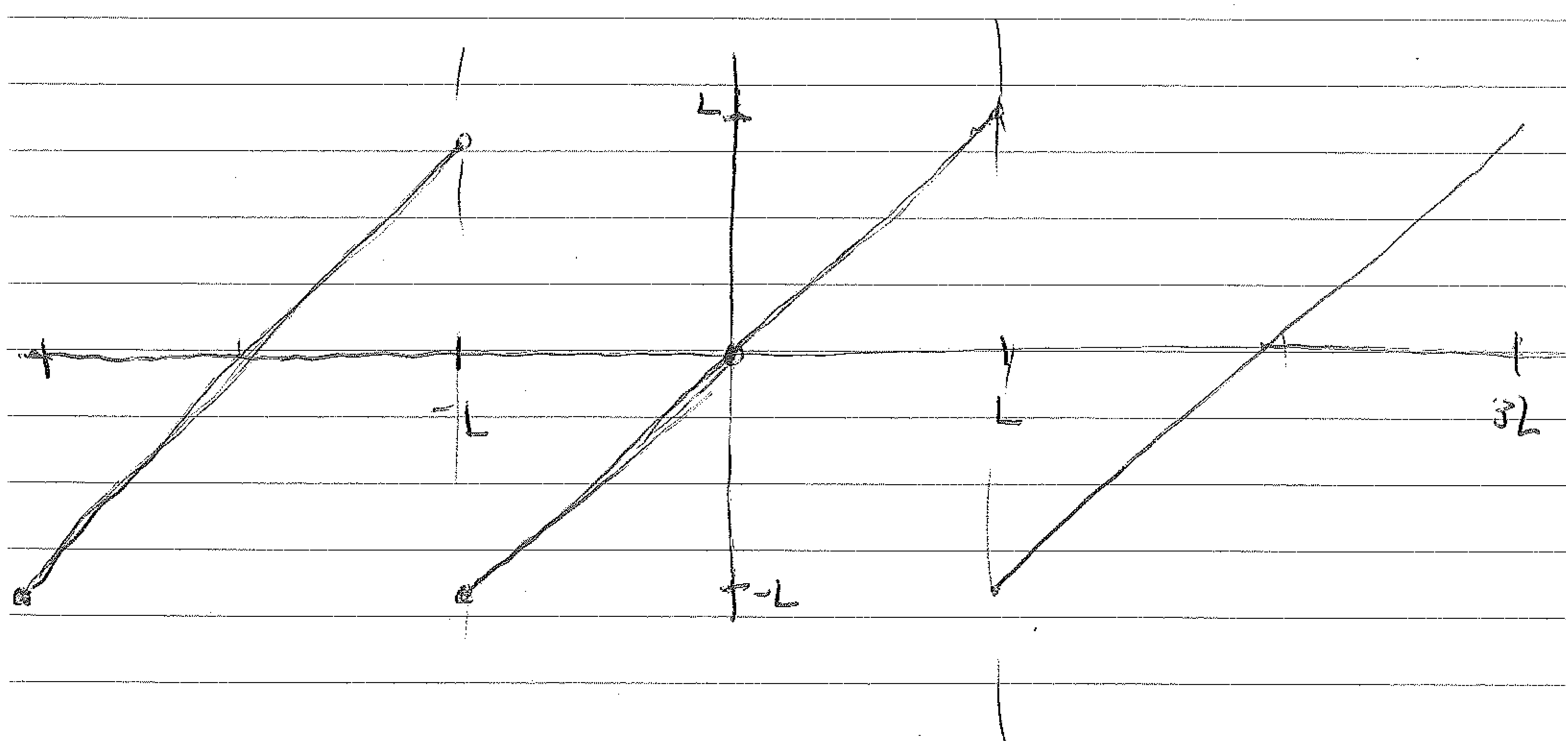
* coefficients that behave like $\frac{1}{n}$ is the slowest converge associated with a jump discontinuity of the function, even at the endpoints $-L$ or L


Gibbs phenomenon for square wave
f(x) = (4/pi) Sum$_1^N$ sin((2k−1)x)/(2k−1)
x
1.5
1
0.5
0
−0.5
−1
−1.5
−4
−3
−2
−1
0
1
2
3
4

* The Gibb's Phenomenon: For a finite number of terms, there is an overshoot of approximately 9% of the jump discontinuity (18% of the half-jump); the overshoot moves closer to the point of discontinuity as N increases (N is the number of terms), but does not disappear.

Example 2: We also expect $\frac{1}{n}$ behavior of the b_n's and a Gibbs phenomenon for

$F(x) = x \quad -L \leq x \leq L$



What really matters is the periodic extension!

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\frac{n\pi x}{L}\, dx \qquad \text{Let } L = \pi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \qquad \text{integrate by parts}$$

$$u = x \qquad dv = \sin nx \, dx$$
$$du = dx \qquad v = \frac{-\cos nx}{n}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n}\, dx \right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi - \pi\cos(-n\pi)}{n} + \frac{\sin nx}{n^2}\Big|_{-\pi}^{\pi} \right]$$

$$= \frac{-\cos n\pi - \cos n\pi}{\pi n} + \frac{1}{\pi}\left[0 - 0\right]$$

$$= -\frac{2\cos n\pi}{\pi n}$$

$$f(x) = x \sim \sum_{n=1}^{\infty} \left(\frac{-2\cos n\pi}{\pi n}\right) \sin nx$$

coefficients again behave like $\frac{1}{n}$

Gibb's phenomenon at $x = -L$, $x = L$