$\boxed{\text{Math 322 Lecture 15}}$

Finish up Chapter 3 $\boxed{\text{Today}}$

* Integration term by term of Fourier Series

* Complex form of Fourier Series

Start Chapter 4 $\boxed{\text{Friday}}$

* The 1D Wave Eqn. and boundary conditions

$\boxed{\text{Integration}}$ of a Fourier series is always ok

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$\int_{-L}^{x} f(x)\,dx = \int_{-L}^{x} a_0\,dt$$

$$+ \sum_{n=1}^{\infty}\left\{a_n \int_{-L}^{x} \cos\frac{n\pi t}{L}\,dt + b_n \int_{-L}^{x} \sin\frac{n\pi t}{L}\,dt\right\}$$

$$= a_0(x+L) + \sum_{n=1}^{\infty}\left\{\frac{L}{n\pi}a_n \sin\frac{n\pi t}{L}\Big|_{-L}^{x} - \frac{L}{n\pi}b_n \cos\frac{n\pi t}{L}\Big|_{-L}^{x}\right\}$$

$$= a_0(x+L) + \sum_{n=1}^{\infty} \frac{L}{n\pi} a_n \sin\frac{n\pi x}{L}$$

$$+ \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n \left(\cos n\pi - \cos\frac{n\pi x}{L}\right)$$

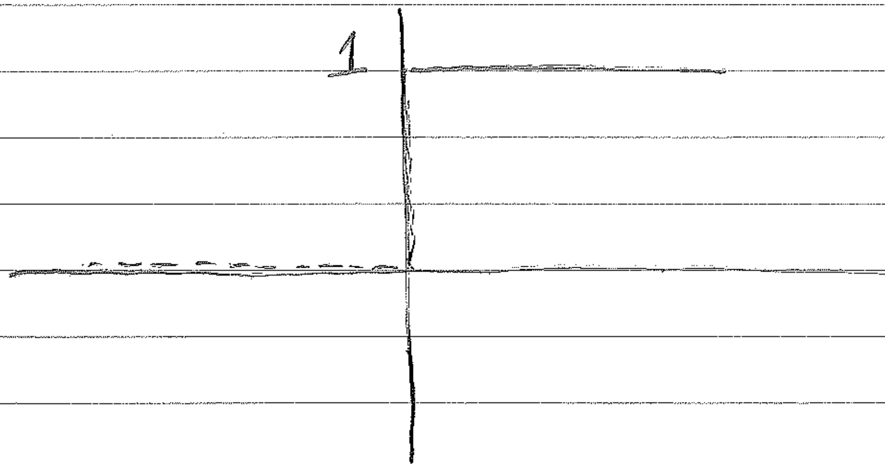
the "=" sign is always ok

Even if the periodic extension of $f(x)$ has jump discontinuities, ~~so~~ the integral will be continuous (integration smooths)

The resulting series is not necessarily a Fourier series because of the $a_0(x+L)$ term

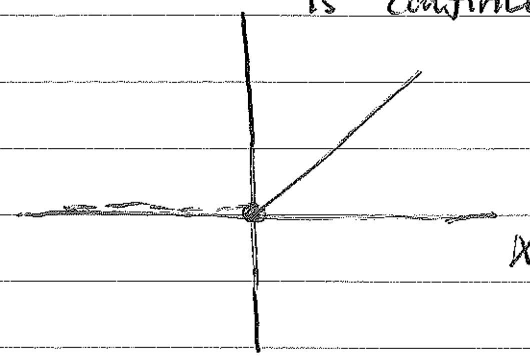
Integration of a jump discontinuity is a "ramp"

e.g.



jump in $f(x)$

$\int_{-\infty}^{x} f(t)\,dt$ is continuous



$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$\int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$$

$(x<0)$ $$\int_{-\infty}^{x} f(t)\,dt = 0 \qquad (x<0)$$

$(x>0)$ $$\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0^-} f(t)\,dt + \int_{0^+}^{x} f(t)\,dt$$

$$= 0 + \int_{0^+}^{x} 1\,dt = x \qquad (x>0)$$

The formal result:

Lets define a $G(x) = F(x) - a_0(x+L)$

where $F(x) = \int_{-L}^{x} f(t)\,dt$

since $F(-L) = 0$ and $F(L) = 2La_0$

$\Rightarrow$ $G(-L) = 0$ $\quad G(L) = 2La_0 - a_0 2L = 0$

So $G(x)$ is continuous and satisfies $G(-L) = G(L) = 0$

$$\Rightarrow G(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}$$

where the = sign is ok!

$A_0 = \ldots$

$A_n = \ldots \quad n \neq 0$

$B_n = \ldots \quad n \neq 0$

and we also have $G(x) = \int_{-L}^{x} f(t)\,dt - a_0(x+L)$

$$= \sum_{n=1}^{\infty} \frac{L}{n\pi} a_n \sin\frac{n\pi x}{L}.$$

$$+ \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n \left(\cos n\pi - \cos\frac{n\pi x}{L}\right)$$

and we need to show

#1

$$n \neq 0 \quad A_n = \frac{1}{L}\int_{-L}^{L} G(x)\cos\frac{n\pi x}{L}\,dx = -\frac{b_n L}{n\pi}$$

[integrate by parts after using the definition of $G(x)$]

#2

$$n \neq 0 \quad B_n = \frac{1}{L}\int_{-L}^{L} G(x)\sin\frac{n\pi x}{L}\,dx = a_n\frac{L}{n\pi}$$

[integrate by parts]
see next page

$$n = 0 \qquad G(L) = 0 \implies$$

$$G(L) = F(L) - 2a_0 L = 0$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{-b_n L}{n\pi}\right)\cos n\pi \qquad \text{from above}$$

$$\implies A_0 = \sum_{n=1}^{\infty}\left(\frac{b_n L}{n\pi}\right)\cos n\pi \qquad \#3$$

e.g. $B_n = \frac{1}{L} \int G(x) \sin\frac{n\pi x}{L}\, dx$

integrate by parts:

$$= \frac{1}{L} \left\{ -G(x) \frac{L}{n\pi} \cos\frac{n\pi x}{L} \Big|_{-L}^{L} + \int_{-L}^{L} \frac{L}{n\pi} G'(x) \cos\frac{n\pi x}{L} dx \right\}$$

$$= \frac{1}{L} \left\{ -G(L) \frac{L}{n\pi} \cos n\pi + G(-L) \frac{L}{n\pi} \cos(-n\pi) \right.$$

$$\left. + \int_{-L}^{L} \frac{L}{n\pi} \left(F'(x) - a_0 \right) \cos\frac{n\pi x}{L}\, dx \right\}$$

with $G(L) = G(-L) = 0$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx = 0$$

$$B_n = \frac{1}{L} \int_{-L}^{L} \frac{L}{n\pi} F(x) \cos\frac{n\pi x}{L} dx = \frac{L}{n\pi} a_n$$

$\boxed{\text{Complex Fourier Series}}$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{2i}\left(e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\right)$$

with $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad \& \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Re-arrange:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)e^{\frac{in\pi x}{L}}$$

$$+ \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)e^{-\frac{in\pi x}{L}}$$

Change $m = -n$ in the ~~[illegible]~~ First sum

$$F(x) \sim a_0 + \sum_{m=-1}^{-\infty} \left(\frac{a_{-m}}{2} + \frac{b_{-m}}{2i} \right) e^{-im\pi x/L}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-in\frac{\pi x}{L}}$$

$$a_{-m} = \frac{1}{L} \int_{-L}^{L} F(x) \cos\left(\frac{-m\pi x}{L} \right) dx$$

$$= \frac{1}{L} \int_{-L}^{L} F(x) \cos \frac{m\pi x}{L} \, dx \quad = \quad a_m$$

$$b_{-m} = \frac{1}{L} \int_{-L}^{L} F(x) \sin\left(\frac{-m\pi x}{L} \right) dx$$

$$= \frac{-1}{L} \int_{-L}^{L} F(x) \sin\left(\frac{m\pi x}{L} \right) dx \quad = \quad -b_m$$

$$F(x) \sim a_0 + \sum_{m=-1}^{-\infty} \left(\frac{a_m}{2} - \frac{b_m}{2i} \right) e^{-im\pi x/L}$$

$$+ \sum_{m=1}^{\infty} \left(\frac{a_m}{2} - \frac{b_m}{2i} \right) e^{-im\pi x/L}$$