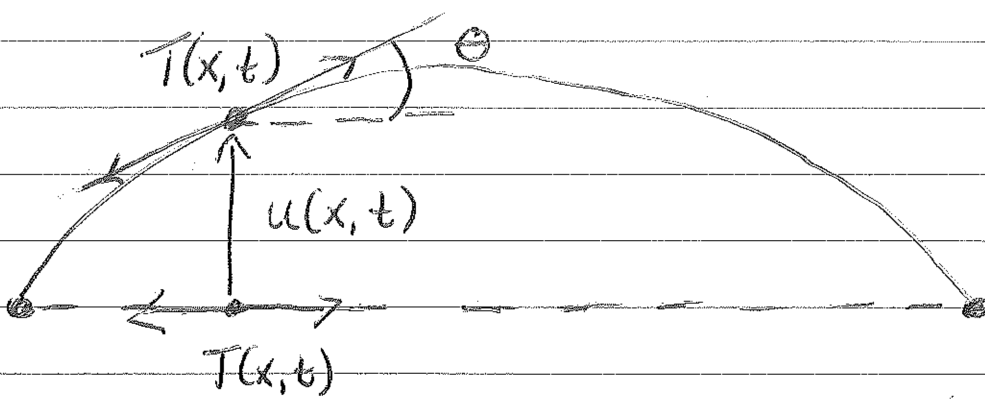
The 1D Wave Equation

Math 322
Lecture 17

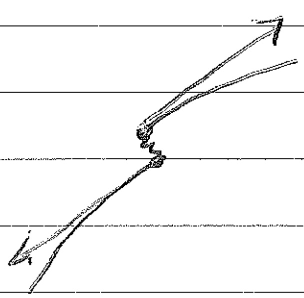
①



Think of a string under tension, e.g. the string of a musical instrument

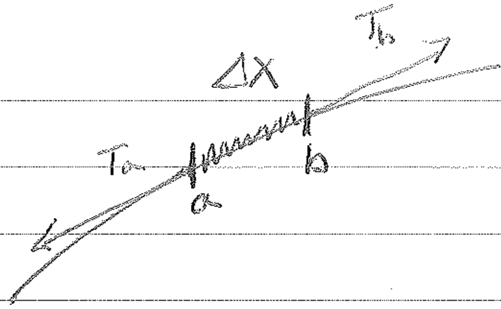
Now make a vertical displacement from the equilibrium position (points on string only move up/down)

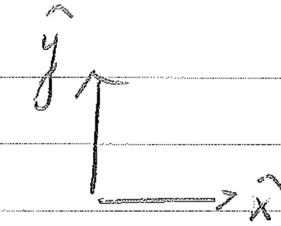
$T(x,t)$ is the <u>magnitude</u> of the tension; the tension is a force that is defined to be tensile.



there are opposite forces pulling from either side, equal in magnitude

$u(x,t)$ is displacement from equilibrium in the <u>vertical</u> direction





Conservation of Momentum in the $\hat{y}$-direction

segment of the curve moving up and down

"Control line" analysis

$$\frac{d^2}{dt^2}\int_a^b \rho(x)\,u(x,t)\,dx = T(b)\sin\theta_b - T(a)\sin\theta_a$$

each segment of the string is like a particle

$\rho(x)$ mass per unit length

$\{ m a = \text{sum of forces} \}$

$$\frac{d^2}{dt^2}\int_a^b \rho(x)u(x,t)\,dx = \int_a^b \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x)\right]dx$$

$$\int_a^b \left\{ \frac{\partial^2}{\partial t^2}\left[\rho(x)\,u(x,t)\right] - \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x)\right] \right\} dx = 0$$

$\forall\ a,b$ {for all a, b} $\Rightarrow$

$$\rho(x)\frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial}{\partial x}\left[T(x,t)\sin\theta(x)\right]$$

Let $\hat{t}$ be the tangent vector

$$\frac{\partial}{\partial x}\left(T(x,t)\sin\theta\right) = \lim_{\Delta x \to 0} \left[\frac{T^{+}\hat{t}^{+} - T^{-}\hat{t}^{-}}{\Delta x}\right] \cdot \hat{y}$$

$$T^{+} = T(x+\Delta x, t) \qquad \hat{t}^{+} = \hat{t}\,(x+\Delta x)$$

$$T^{-} = T(x,t) \qquad \hat{t}^{-} = \hat{t}\,(x)$$

$$\hat{t} = \frac{\cos\theta\,\hat{x} + \sin\theta\,\hat{y}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{1\hat{x} + \frac{\partial u}{\partial x}\hat{y}}{\sqrt{1^2 + \left(\frac{\partial u}{\partial x}\right)^2}}$$ a unit vector

$$\hat{t}\cdot\hat{y} = \frac{\frac{\partial u}{\partial x}}{\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}} = \sin\theta$$

Approximations

$$\left.\begin{array}{l} ① \quad T(x,t) = T_0 \\ ② \quad \hat{t}\cdot\hat{y} \approx \frac{\partial u}{\partial x} \end{array}\right\}$$ consistent with

small vertical displacements / small angles

and $\left(\frac{\partial u}{\partial x}\right)^2 \ll 1$

So we arrive at

$$\rho(x) \frac{\partial^2}{\partial t^2} u(x,t) \approx \frac{\partial}{\partial x}\left(T_0 \frac{\partial u}{\partial x} \right)$$

$$\frac{\partial^2}{\partial t^2} u(x,t) \approx T_0/\rho(x) \frac{\partial^2}{\partial x^2} u(x,t)$$

with $T_0/\rho(x) > 0 \Rightarrow$

$$\frac{\partial^2}{\partial t^2} u(x,t) \approx c^2 \frac{\partial^2}{\partial x^2} u(x,t)$$

$T_0 > 0$ means the string is always under tension

Now what are sensible boundary conditions?

$\rho(x) = \rho_0$ constant $\Rightarrow$ $c^2 = T_0/\rho_0$
constant

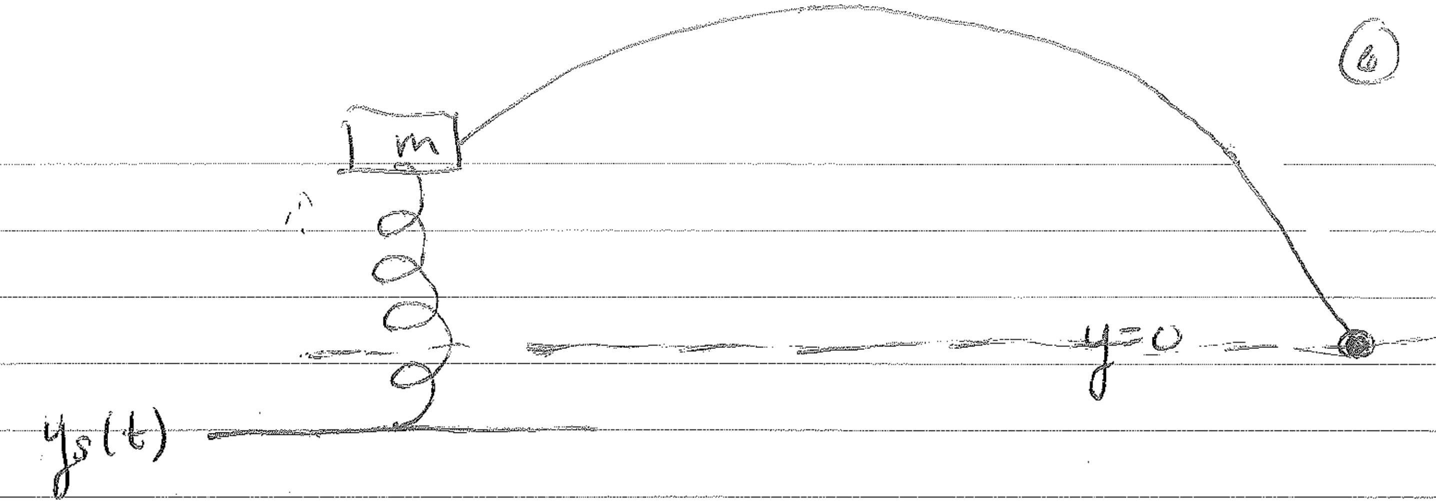
(1) Prescribed values of the displacement at the ends

$$u(0,t) = f(t) \qquad u(L,t) = g(t)$$

(2) Vanishing vertical component of the tensile force at the ends

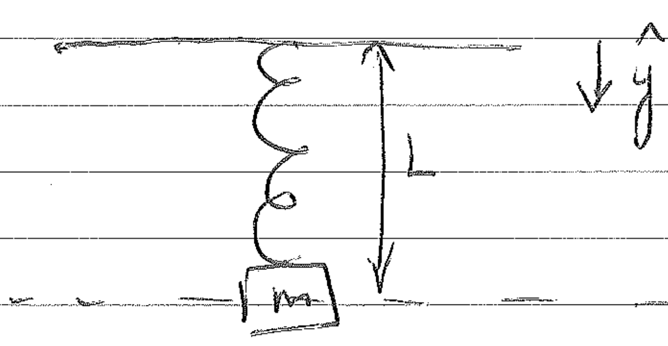
$$T_0 \frac{\partial u}{\partial x}\Big|_{x=0} = 0 \qquad T_0 \frac{\partial u}{\partial x}\Big|_{x=L} = 0$$

(3) A restoring force at the ends $\Rightarrow$ the analog of Newton's Law of Cooling



$y_s(t)$ is the vertical position of the support (can be moving in time in some prescribed way)

Start with what we understand:



mass on a spring under gravity

L = equilibrium spring length

✓ Equilibrium: $(0 = mg - kL)\hat{y}$

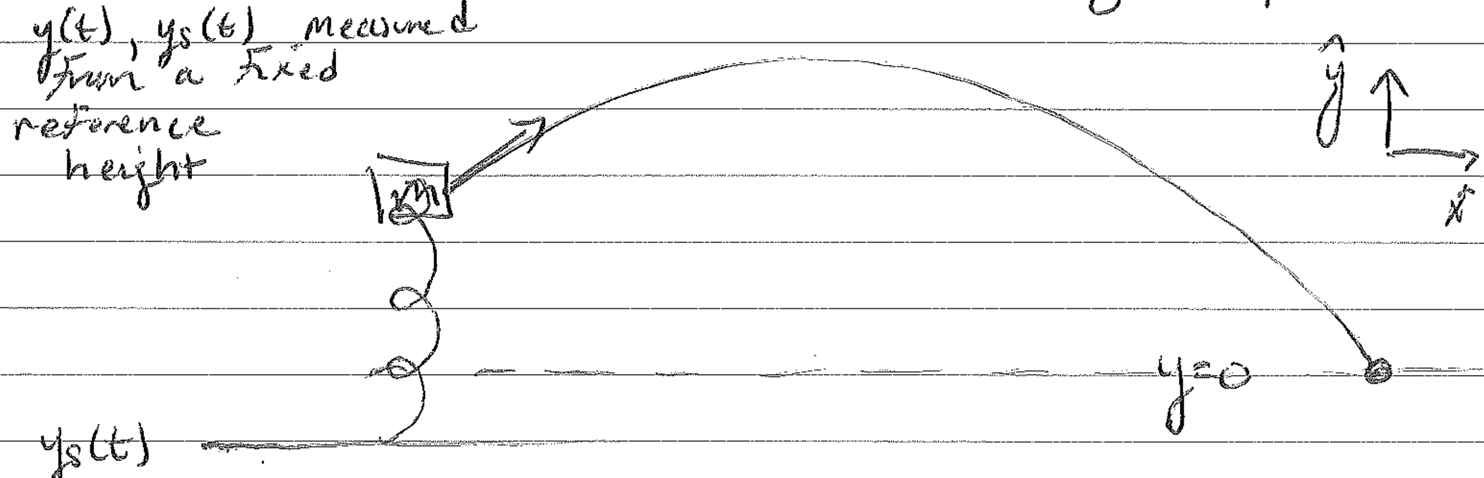
k is the spring constant

$\underline{F} = -kL\hat{y}$ is the restoring force

$\boxed{\text{In motion}}$ $$m\frac{d^2y(t)}{dt^2} = mg - k\left[y(t) - L\right]$$

$y(t)$ is measured from the support, assuming that the support does not move

Now flip upside-down; allow the support to move; add the force due to the string (rope)



$$m\frac{d^2y(t)}{dt^2} = -mg - k\left[y(t) - \left(y_s(t) + L\right)\right]$$

$$+ T_0 \frac{\partial u}{\partial x}\Big|_{x=0}$$

$$\left\{\frac{\partial u}{\partial x} > 0\right\}$$ in picture

extra vertical force up

$y - y_s > L \Rightarrow$ spring is stretched

$y - y_s < L \Rightarrow$ spring is compressed

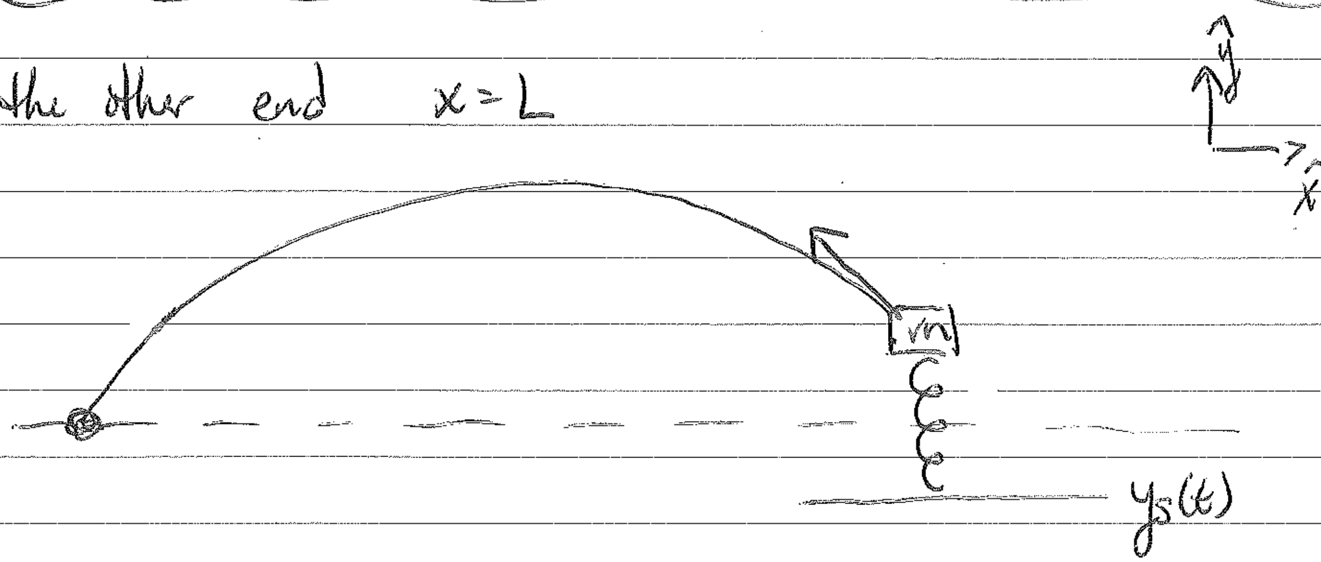
Now "go massless"

$$T_0 \frac{\partial u}{\partial x}\bigg|_{x=0} = k\left[y(t) - (y_S(t)+L)\right]$$

$$= k\left[u\big|_{x=0} - u_E(t)\right]$$

$u_E(t) = u_{END}(t)$ prescribed

At the other end $x = L$



$$m\frac{d^2y(t)}{dt^2} = -mg - k\left[y(t) - (y_S(t)+L)\right]$$

$$+ T_0\left(-\frac{\partial u}{\partial x}\right)\bigg|_{x=L}$$

extra force up

$\left\{\frac{\partial u}{\partial x} < 0\right\}$

in picture

Now "go massless"

$$-T_0 \frac{\partial u}{\partial x}\bigg|_{x=L} = k\left[u\big|_{x=L} - u_E(t)\right]$$