$\boxed{\text{Math 322 Lecture 19}}$ Start Ch. 5

Regular Sturm Liouville Problems:

$$L(\phi) = -\lambda \sigma(x) \phi \qquad \phi = \phi(x) \qquad a < x < b$$

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$$

$p(x), q(x), \sigma(x)$ real continuous $a \le x \le b$
(including endpoints)

$p(x) > 0$, $\sigma(x) > 0$ in $a \le x \le b$

Boundary Conditions

$$\beta_1 \phi(a) + \beta_2 \left.\frac{d\phi(x)}{dx}\right|_{x=a} = 0$$

$$\beta_3 \phi(b) + \beta_4 \left.\frac{d\phi(x)}{dx}\right|_{x=b} = 0$$

[1] The operator together with the boundary conditions is symmetric

[also called Hermitian or Self-adjoint]

Notation: $(Lu, v) = (u, Lv)$

For real-valued functions satisfying the b.c.s

$$\int_a^b v L(u)\, dx = \int_a^b u L(v)\, dx$$

For complex-valued functions satisfying the b.c.s

$$\int_a^b \bar{v} L(u)\, dx = \int_a^b u L(\bar{v})\, dx$$

$$(Lu, \bar{v}) = (u, L\bar{v})$$

where bar is complex conjugate

Let's show this for real-valued functions using integration by parts

$$\int_a^b u L(v)\, dx = \int_a^b \left[(pv')' u + quv \right] dx$$

$s = u \qquad dt = (pv')' dx$
$ds = u' dx \qquad t = pv'$

$$= upv' \Big|_a^b - \int_a^b u' p v' \, dx + \int_a^b quv \, dx$$

$\downarrow$

by parts again $\qquad s = -pu' \qquad dt = v' dx$
$ds = -(pu')' dx \qquad t = v$

$$= upv' \Big|_a^b - vpu' \Big|_a^b + \int_a^b v (pu')' dx + \int_a^b quv \, dx$$

$$= \int_a^b v L(u)\, dx + p \left[uv' - vu' \right] \Big|_a^b$$

$$\Rightarrow \int_a^b \left[uL(v) - vL(u) \right] dx = p \left[uv' - vu' \right] \Big|_a^b$$

"Green's Formula"

Note that Green's Formula can be written

$$\int_a^b \left[uL(v) - vL(u) \right] dx = p\left[uv' - vu' \right]\Big|_a^b$$

$$= \int_a^b \frac{d}{dx}\left[p\left(uv' - vu' \right) \right] dx$$

leading to

$$uL(v) - vL(u) = \frac{d}{dx}\left[p\left(uv' - vu' \right) \right]$$

"the differential form of Lagrange's Identity"

Now assume that u, v both satisfy the RSL problem; same boundary conditions

$$L(u) = -\lambda \sigma u \qquad L(v) = -\lambda \sigma v$$

$$\beta_1 u\Big|_a + \beta_2 \frac{du}{dx}\Big|_a = 0 \qquad \beta_1 v\Big|_a + \beta_2 \frac{dv}{dx}\Big|_a = 0$$

$$\beta_3 u\Big|_b + \beta_4 \frac{du}{dx}\Big|_b = 0 \qquad \beta_3 v\Big|_b + \beta_4 \frac{dv}{dx}\Big|_b = 0$$

Plug in to find

$$\int_a^b \left[u L(v) - v L(u) \right] dx = 0$$

where we have used

* $p(x)$ continuous in $a \le x \le b$
* boundary conditions

$$u'(a) = -\frac{\beta_1}{\beta_2} u(a) \qquad v'(a) = -\frac{\beta_1}{\beta_2} v(a)$$

$$u'(b) = -\frac{\beta_3}{\beta_4} u(b) \qquad v'(b) = -\frac{\beta_3}{\beta_4} v(b)$$

For complex-valued functions, we need to split into real and imaginary parts and continue ...

Do this as an exercise !

[2] The eigenvalues are real

Suppose λ is complex and $\phi(x)$ is complex:

$\lambda = \mu + i\nu$, $\phi(x) = U(x) + iV(x)$

The equation is

$$L(\phi) = -\lambda \sigma \phi$$

we can take the complex conjugate to find

$L(\bar{\phi}) = -\bar{\lambda} \sigma \bar{\phi}$ since $\sigma(x)$ assumed real

Now by symmetry of the operator / b. c.'s

$$\int_a^b \bar{\phi} L(\phi)\, dx = \int_a^b \phi L(\bar{\phi})\, dx$$

$$\int_a^b \bar{\phi}(-\lambda \sigma \phi)\, dx = \int_a^b \phi(-\bar{\lambda} \sigma \bar{\phi})\, dx$$

$$\phi \bar{\phi} = (U + iV)(U - iV) = U^2 + V^2$$

$\sigma(x) > 0$ by assumption

$$(\lambda-\bar{\lambda})\int_a^b \sigma\,\phi\bar{\phi}\,dx = 0$$

(and continuous)

Since $\sigma\phi\bar{\phi} > 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$= 2i\nu$

$\Rightarrow \lambda$ real

~~STOP HERE~~

Next Class

$\boxed{3}$ For each eigenvalue, there is only one linearly independent eigenfunction

[not true for periodic boundary conditions!]

Assume $L(\phi_1) = \lambda_0\sigma\phi_1 \qquad L(\phi_2) = \lambda_0\sigma\phi_2$

with ϕ_1, ϕ_2 linearly independent for $\lambda = \lambda_0$;

and ϕ_1, ϕ_2 satisfy the boundary conditions

By definition of linear independence on $a \le x \le b$

$$\left.\begin{aligned} c_1\phi_1(x) + c_2\phi_2(x) &= 0 \\ c_1\phi_1'(x) + c_2\phi_2'(x) &= 0 \end{aligned}\right\} \quad a \le x \le b$$