# Math 322 Lecture 2

Conservation of Energy (internal) in an arbitrary volume of solid

"Control Volume" analysis

$$\frac{d}{dt}\int_{\forall} e \, d\forall = \oint_{A} \underset{\sim}{q} \cdot (-\hat{n})\, dA + \int_{\forall} Q \, d\forall$$

Each term in this eqn. has dimensions of energy/time or $\left[\frac{m\ell^2}{t^3}\right]$

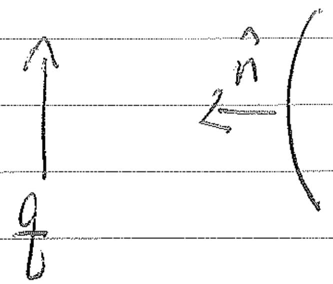
energy $[E] = \left[\frac{m\ell^2}{t^2}\right]$

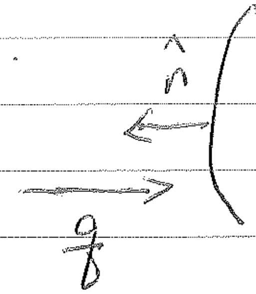
energy density $[e] = \left[\frac{m}{\ell t^2}\right]$

All quantities are functions of 3 space dimensions and time: $e(\underset{\sim}{x}, t)$, $\underset{\sim}{q}(\underset{\sim}{x}, t)$, $Q(\underset{\sim}{x}, t)$

$\underset{\sim}{q}(\underset{\sim}{x}, t)$ is the energy flux; $\underset{\sim}{q}$ is a vector; $\underset{\sim}{q} \cdot (-\hat{n})$ is a scalar

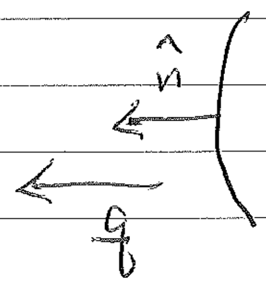
"Easy" cases to think about




$q \cdot (-\hat{n}) = 0$ $\qquad\qquad$ $q \cdot (-\hat{n}) > 0$

$\Rightarrow$ "energy in"




$q \cdot (-\hat{n}) < 0 \quad \Rightarrow$ "energy out"

## Dimensional Analysis

$\oint_A q \cdot (-\hat{n})\, dA$ must have $\left[\dfrac{m\ell^2}{t^3}\right]$

$$\Rightarrow \quad [q] = \left[\frac{m}{t^3}\right]$$

$\int_{\forall} Q \, d\forall$ must be $\left[\dfrac{m\ell^2}{t^3}\right]$

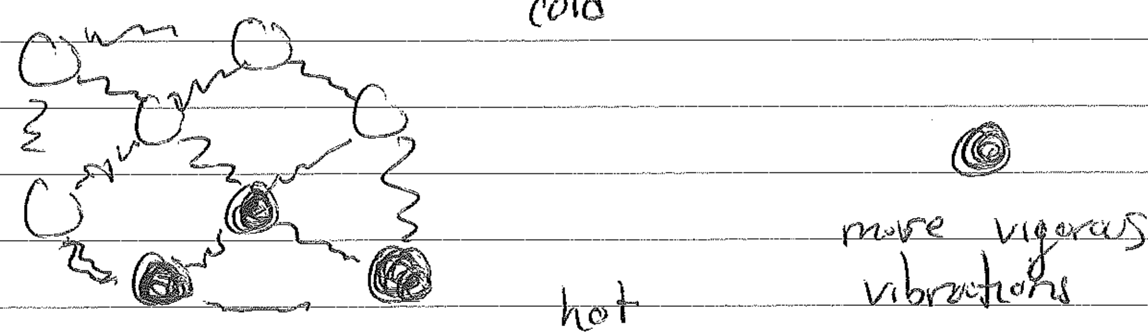
$$\Rightarrow \quad [Q] = \left[\frac{m}{\ell t^3}\right]$$

How do we write energy conservation as a temperature equation?

1. Internal energy depends on

(i) density of the solid material (how much material after integration over volume)

(ii) temperature (a measure of equilibrium)




$$\Longrightarrow \quad e(\underline{x},t) \propto \rho(\underline{x},t)\, T(\underline{x},t)$$

$$\left[\frac{m}{\ell t^2}\right] \qquad \left[\frac{m}{\ell^3}\right] \quad \left[{}^\circ K\right]$$

hence there must be a proportionality coefficient

$$c(\underline{x},t) \quad \text{with} \quad [c] = \left[\frac{\ell^2}{t^2\, {}^\circ K}\right]$$

Physical meaning of $c(\underline{x}, t)$ :

Compare

$$[E] = \left[\frac{m\,l^2}{t^2}\right] \quad \text{to} \quad [c] = \left[\frac{l^2}{t^2\,{}^\circ K}\right]$$

$c(\underline{x}, t)$ represents the amount of energy needed to raise the temperature of a unit mass by $1^\circ K$.

$c$ is called the "specific heat coefficient"; a thermodynamic variable; function of 2 of $\rho, p, T$

e.g. $c(\rho, T) = c(\underline{x}, t)$

So $e(\underline{x}, t) = c(\underline{x}, t)\,\rho(\underline{x}, t)\,T(\underline{x}, t)$

2. What is $\underline{q}$ ?

Fourier's law (an experimental fact) : energy flows from hot to cold

$\underline{q} \propto -\underline{\nabla} T$ ; $\underline{\nabla} T$ is in the direction from cold to hot

$$\underline{q}(\underline{x},t) = -k(\underline{x},t)\,\underline{\nabla}\,T(\underline{x},t)$$

$$\left[\frac{m}{t^3}\right] \qquad \left[\frac{m\ell}{t^3\,{}^\circ K}\right]\left[\frac{1}{\ell}\right]\left[{}^\circ K\right]$$

$$\Rightarrow \quad [k] = \left[\frac{m\ell}{t^3\,{}^\circ K}\right] \quad \text{"thermal conductivity"}$$

Finally we arrive at

$$\frac{d}{dt}\int_{\forall} c\rho T\, d\forall = \oint_{A} k\underline{\nabla}T\circ\hat{n}\, dA + \int_{\forall} Q\, d\forall$$

This is now conservation of energy in control volume form in terms of temperature $T$

Next:
- Use Divergence Thm
- Shrink the volume to a point
- Arrive at the PDE
- Simplify the PDE

Divergence Thm:

$$\int_{\forall} \underline{\nabla} \cdot \underline{F} \, d\forall = \oint_{A} \underline{F} \cdot \hat{n} \, dA$$

where $\underline{F} = \underline{F}(\underline{x})$ is a vector field in 3D

$$\left\{ \begin{array}{l} \text{a generalization of the Fundamental Thm of} \\ \quad \text{Calculus in 1D} \\ \quad \displaystyle\int_a^b \frac{\partial F}{\partial x} \, dx = \underbrace{F(b) - F(a)}_{\text{boundary terms}} \end{array} \right\}$$

$\Rightarrow$

$$\frac{d}{dt} \int_{\forall} c\rho T \, d\forall = \int_{\forall} \underline{\nabla} \cdot \left[ k \, \underline{\nabla} T \right] d\forall + \int_{\forall} Q \, d\forall$$

all volume integrals; all scalar quantities / integrands

** $\nabla \cdot \underline{V}$ arises from (i) a conservation law; (ii) a flux law; (iii) divergence thm

Now shrink $V$ to a point:

$$\frac{\partial}{\partial t}(c\rho T) = \underline{\nabla} \cdot [k \underline{\nabla} T] + Q$$

where each term has dimensions $\left[\frac{\text{energy}}{\text{volume} \cdot \text{time}}\right]$

$c(\underline{x},t)$ $\rho(\underline{x},t)$ $T(\underline{x},t)$ $k(\underline{x},t)$ $Q(\underline{x},t)$

Simplify: For a constant density material in a restricted temperature range (no explosions)

$\rho = \rho_0$, $c(\rho,T) \approx c_0$, $k(\rho,T) \approx k_0$ $\Rightarrow$

$$c_0\rho_0 \frac{\partial T}{\partial t} = k_0 \nabla^2 T + Q \quad \text{or}$$

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \tilde{Q}$$

$$\kappa = \frac{k_0}{c_0\rho_0} \qquad \tilde{Q} = \frac{Q}{c_0\rho_0} \qquad (\text{drop } \sim)$$