Math 322 Singular SL problems and why the nice results ... { Bessel, Legendre, Chebyshev }

$L[\phi] = -\lambda\sigma\phi \quad \phi = \phi(x) \quad a < x < b$

$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$

$p(x),\ q(x),\ \sigma(x)$ real continuous in $a \le x \le b$

What if we relax the conditions

$p(x),\ \sigma(x) > 0$ at one (or both) endpoints

and allow boundedness condition at the same points

e.g. Bessel's Equation of Order Zero

$-(xy')' = \lambda x y \qquad 0 < x < 1$

$y(0), y'(0)$ bounded $\qquad y(1) = 0$

Most of the nice results rely on symmetry, so if we can establish symmetry, then we are well on our way

Bessel's Eqn. has $p(0)=0$, $\sigma(0)=0$ and boundedness at $x=0$

Lets do a more general analogous problem

$$L[\phi] = -\lambda\sigma\phi \qquad a<x<b$$

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$$

$p(x)$, $q(x)$, $\sigma(x)$ real continuous $a \le x \le b$

$p(a) = \sigma(a) = 0$ otherwise positive $a < x \le b$

$\phi(a)$, $\phi'(a)$ bounded

$$\beta_3\phi(b) + \beta_4\phi'(b) = 0$$

To establish symmetry, we need to show
(real functions to reduce calculations)

$$\int_a^b u\,L[v]\,dx = \int_a^b v\,L[u]\,dx$$

Integration by parts $\Rightarrow$

$$\int_a^b uL[v]\,dx = \int_a^b vL[u]\,dx + p[uv' - vu']\Big|_a^b$$

Lets evaluate $p[uv' - vu']\Big|_a^b$

$$= p(b)\left[u(b)v'(b) - v(b)u'(b)\right] - p(a)\left[u(a)v'(a) - v(a)u'(a)\right]$$

$$= p(b)\left[u(b)\left(-\frac{\beta_3}{\beta_4}\right)v(b) - v(b)\left(-\frac{\beta_3}{\beta_4}\right)u(b)\right]$$

$$- 0\left[\underbrace{u(a)v'(a)}_{\text{both bounded}} - \underbrace{v(a)u'(a)}_{\text{both bounded}}\right]$$

$$\Rightarrow \int_a^b uL[v]\,dx = \int_a^b vL[u]\,dx$$

Math 322 Lecture 21

The Rayleigh Quotient

The SL Equation (regular or singular)

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]\phi(x) + q(x)\phi(x) = -\lambda\sigma(x)\phi(x)$$

Multiply by $\phi(x)$ and integrate:

$$\int_a^b \left\{\phi\frac{d}{dx}\left[p\frac{d\phi}{dx}\right] + q\phi^2\right\}dx = -\lambda\int_a^b \sigma\phi^2\,dx$$

Solve for $\lambda = \dfrac{-\int_a^b\left(\phi\frac{d}{dx}\left[p\frac{d\phi}{dx}\right] + q\phi^2\right)dx}{\int_a^b \sigma\phi^2\,dx}$

What can we learn about λ from this integral expression?

Can we tell if $\lambda \geq 0$ as we've seen in our heat conduction problems and a wave eqn problem [often $\lambda \geq 0$ is physical]

Lets message a little using integration by parts

$$\int_a^b \phi \frac{d}{dx}\left[p \frac{d\phi}{dx}\right] dx \qquad s = \phi \quad dt = (p\phi')' dx$$
$$ds = \phi' dx \quad t = p\phi'$$

$$= p\phi\phi'\Big|_a^b - \int_a^b p(\phi')^2 dx \implies$$

$$\lambda = \frac{-p\phi\phi'\Big|_a^b + \int_a^b p(\phi')^2 dx - \int_a^b q\phi^2 dx}{\int_a^b \sigma \phi^2 dx}$$

called the Rayleigh quotient

Now we can see that $\lambda \geq 0$ if

(a) $-p\phi\phi'\Big|_a^b \geq 0$

(b) $q \leq 0$

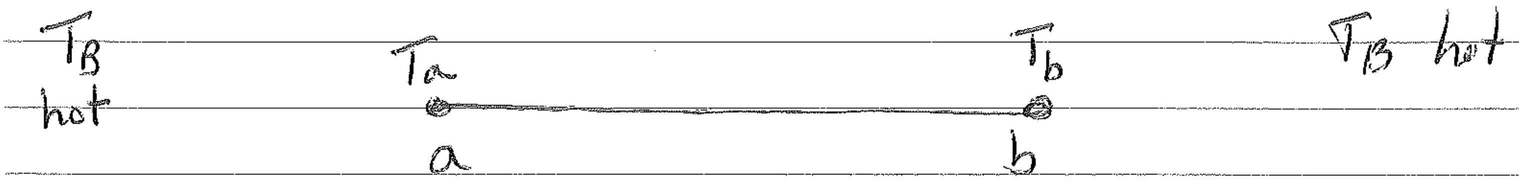
We've seen $\lambda \geq 0$ in heat conduction problems and a wave-equation problem (often $\lambda \geq 0$ is physical)

From the Rayleigh quotient, one can see that $\lambda \geq 0$ if

(a) $-p\phi\phi'\Big|_a^b \geq 0$ and

(b) $q \leq 0$

(a) Lets think about 1D heat eqn.



$$k \frac{dT}{dx}\Big|_a = H(T_a - T_B) \qquad k\frac{dT}{dx}\Big|_b = H(T_B - T_b)$$

But for a regular Sturm Liouville problem we need homogeneous boundary conditions

$$k \frac{dT}{dx}\Big|_a = H T_a \qquad k\frac{dT}{dx}\Big|_b = -H T_b$$

$$-p\phi_b\phi_b' - (-p\phi_a\phi_a')$$

$$= -p\phi_b(-H\phi_b) - (-p\phi_a H\phi_a)$$

$$= Hp\phi_b^2 + Hp\phi_a^2 \geq 0$$

$$\{H > 0,\ p > 0,\ \phi_b^2 \geq 0,\ \phi_a^2 \geq 0\}$$

(b) $q \leq 0$ is also physical, e.g.

$$\frac{\partial T}{\partial t} = k\frac{d^2T}{dx^2} - \alpha T$$

is an energy absorbing situation

Unless we know $\phi(x)$, the Rayleigh Quotient doesn't tell us explicitly λ, but can be used to estimate λ

The lowest eigenvalue λ_1 must be the minimum value of the Rayleigh quotient

$$\lambda_1 = \min \frac{\left\{ -p\phi\phi' \Big|_a^b + \int_a^b \left(p(\phi')^2 - q\phi^2 \right) dx \right\}}{\int_a^b \sigma \phi^2 \, dx}$$

assuming ϕ continuous and satisfying the boundary conditions

If we cannot do this minimization problem, we can still get an estimate from above (an overestimate) by using a trial function approach

Consider trial functions $u_T(x)$ that are continuous and satisfy the boundary conditions:

$$\lambda_1 \leq RQ[u_T]$$

Why is the smallest value of λ important?

e.g.

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \qquad 0 < x < L$$

$$T(0,t) = T(L,t) = 0$$

$$T(x,0) = f(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \exp\left[-k\frac{n^2\pi^2}{L^2}t\right]$$

$\lambda_1 = \frac{n^2\pi^2}{L^2}$ is the slowest decay rate

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}$$

Rayleigh-Ritz for Bessel's Eqn. of order zero

$$-(x y')' = \lambda x y \qquad 0 < x < 1$$

$$y(0),\ y'(0) \text{ bounded} \qquad y(1) = 0$$

This eqn. + boundary conditions defines the Bessel function $J_0(\sqrt{\lambda}x)$ of order zero of the 1^{st} kind ; it is a singular SL problem because $p(0) = \sigma(0) = 0$, but it is self-adjoint and it has all the "nice" properties of self-adjoint problems.

Lets make a guess for the eigenfunction corresponding to the lowest eigenvalue, and then find an upper bound for λ_1

From integration by parts :

$$\lambda_1 \le \frac{-p u_T u_T \Big|_a^b + \int_a^b \left\{ p (u_T')^2 - q u_T^2 \right\} dx}{\int_a^b \sigma u_T^2 \, dx}$$

where u_T is a trial function satisfying the b. c.s

What is a "smart" trial function?

* eigenfunctions corresponding to the lowest eigenvalue do not have interior zeros

* $u_T(x)$ satisfies

$u_T(0)$, $u_T'(0)$ bounded , $u_T(1) = 0$

Perhaps: $u_T(x) = \cos \frac{\pi x}{2}$

or $u_T(x) = 1 - x^2$

{ I cheated bc. I know $J_0'(0) = 0$ }

Hint: Use the equivalent of the Rayleigh quotient in terms of the original operator (simplifies the algebra)

Note: the actual value of $\sqrt{\lambda_1} = 2.405$

$$\lambda_1 \le \frac{-\int_0^1 \phi \frac{d}{dx}\left[p \frac{d\phi}{dx}\right] dx}{\int_0^1 x\,\phi^2\,dx}$$

$$\lambda_1 \le \frac{-\int_0^1 \cos\frac{\pi x}{2}\left[-x\frac{\pi}{2}\sin\frac{\pi x}{2}\right]' dx}{\int_0^1 x\cos^2\frac{\pi x}{2}\,dx}$$

$$\lambda_1 \le \frac{\int_0^1 \left\{\frac{\pi}{2}\sin\frac{\pi x}{2}\cos\frac{\pi x}{2} + \left(\frac{\pi}{2}\right)^2 x\cos^2\frac{\pi x}{2}\right\} dx}{\int_0^1 x\cos^2\frac{\pi x}{2}\,dx}$$

Try again for $u_T(x) = 1 - x^2$ . . .

Now how do we find the true λ_1 ?