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SL for large λ (Section 5.9), $\lambda \to \infty$

$$[p\phi']' + q\phi = -\lambda\sigma\phi \quad , \quad \lambda \to \infty \quad , \text{ or}$$

$$\frac{1}{\lambda}[p\phi']' + \frac{1}{\lambda}q\phi + \sigma\phi = 0$$

with $\varepsilon^2 = \frac{1}{\lambda} \to 0$ and the power 2 is for convenience. {It is easier in mathematics to deal with a small parameter; then think about a perturbation expansion approach}

WKB: a technique to analyze linear ODES with a small parameter in front of the highest derivative

Let $\phi(x) \sim \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right]$

where $\delta = \varepsilon^p$ p real, n integer. The parameter p needs to be determined as part of the solution along with the functions $S_n(x)$

The Geometrical Optics approximation:

$$\phi(x) \sim \exp\left[\frac{1}{\delta} S_0(x)\right]$$

The Physical Optics approximation:

$$\phi(x) \sim \exp\left[\frac{1}{\delta} S_0(x) + S_1(x)\right]$$

In many (but not all!) problems $\rho = 1$

For the sake of simplicity, lets think $\rho = 1$ for the rest of the discussion.

$\boxed{\text{Motivation}}$ How can we motivate this expansion?

Go back to $[p\phi']' + q\phi = -\lambda\sigma\phi$

Which term on the LHS balances the RHS for $\lambda \to \infty$ (try subsets of terms)

(i) $q\phi \sim \lambda\sigma\phi \Rightarrow q \sim \lambda\sigma$
not sensible

(ii) $[p\phi']' \sim -\lambda\sigma\phi$

This might work, and if ρ, σ are constants, the approximate solution "looks" like complex exponential : $\rho = \rho_0$, $\sigma = \sigma_0$

$$\phi \sim C_1 \exp\left[i\sqrt{\lambda}\sqrt{\sigma_0/\rho_0}\; x\right] + C_2 \exp\left[-i\sqrt{\lambda}\sqrt{\frac{\sigma_0}{\rho_0}}\; x\right]$$

Then with $\sqrt{\lambda} = \varepsilon^{-1}$, this motivates

$$\phi(x) \sim \exp\Big[\, \varepsilon^{-1} S_0(x) + \underbrace{S_1(x) + \varepsilon S_2(x) + \cdots}_{\text{correction terms}} \,\Big]$$

$S_0(x)$ complex

Formally, we plug in

$$\phi(x) \sim \exp\left[\frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n S_n(x)\right]$$

Lets try it!

$$\phi'(x) \sim \left[\frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n S_n'(x)\right] \exp\left[\frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n S_n(x)\right]$$

$$\phi''(x) \sim \left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n''(x)\right] \exp\left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n(x)\right]$$

$$+ \left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n'(x)\right]^2 \exp\left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n(x)\right]$$

Notice that the common factor of $\exp\left[\dots\right]$ will cancel out when we plug into the eqn. since the equation is linear, homogeneous.

Plug into $\varepsilon^2\left[p\phi'\right]' + \varepsilon^2 q\phi + \sigma\phi = 0$ or

$$\varepsilon^2 p\phi'' + \varepsilon^2 p'\phi' + \varepsilon^2 q\phi + \sigma\phi = 0 \implies$$

$$\varepsilon^2 p(x)\left\{\left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n''(x)\right] + \left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n'(x)\right]^2\right\}$$

$$+ \varepsilon^2 p'(x)\left\{\left[\frac{1}{\varepsilon}\sum_{n=0}^{\infty} \varepsilon^n S_n'(x)\right]\right\}$$

$$+ \varepsilon^2 q(x) \sim -\sigma$$

Now balance the biggest terms which are $O(1)$

$$(S_0')^2 \sim -\frac{\sigma}{\rho} \; ; \quad S_0' \sim \pm i\sqrt{\sigma/\rho}$$

$$S_0(x) \sim \pm i \int \sqrt{\frac{\sigma(x)}{\rho(x)}}\, dx$$

Then we have the Geometrical Optics approx:

$$\phi(x) \approx C_1 \exp\left[i/\varepsilon \int \sqrt{\frac{\sigma(x)}{\rho(x)}}\, dx \right]$$

$$+ C_2 \exp\left[-i/\varepsilon \int \sqrt{\frac{\sigma(x)}{\rho(x)}}\, dx \right]$$

and now we can correct this by finding the next term $S_1(x)$ from $O(\varepsilon)$ terms, and so on.

Find $S_1(x) \sim -1/4 \ln(\sigma(x)\rho(x))$

and the Physical Optics approx:

$$\phi(x) \sim C_1 \exp\left[i/\varepsilon \int (\sigma/\rho)^{1/2} dx - 1/4 \ln(\sigma\rho) + \cdots \right]$$

$$+ C_2 \exp\left[-i/\varepsilon \int (\sigma/\rho)^{1/2} dx - 1/4 \ln(\sigma\rho) + \cdots \right]$$

$$\phi(x) \sim C_1 (\sigma p)^{-1/4} \exp\left[i\sqrt{\lambda} \int \left(\frac{\sigma}{p}\right)^{1/2} dx \right]$$

$$+ C_2 (\sigma p)^{-1/4} \exp\left[\bar{i}\sqrt{\lambda} \int \left(\frac{\sigma}{p}\right)^{1/2} dx \right]$$

and now we need to use the boundary conditions to find the values of λ; we can also use orthogonality as needed.

[Example from Bender & Orszag]

$$y'' = -\lambda (x+\pi)^4 y \qquad y(0) = y(\pi) = 0$$

$$p(x) = 1, \quad q(x) = 0, \quad \sigma(x) = (x+\pi)^4$$

$$\lambda_n \to \infty, \quad 1/\lambda_n = \varepsilon^2 \to 0$$

$$\varepsilon^2 y'' + (x+\pi)^4 y = 0 \qquad y(0) = y(\pi) = 0$$

Let $y(x) \sim \exp\left[\frac{1}{\varepsilon} \sum_{n=0}^{\infty} \varepsilon^n S_n(x) \right]$

Plug in; we've already done the work!

at $O(1)$: $\quad (S_0')^2 \sim -(x+\pi)^4$

$$S_0' \sim \pm i(x+\pi)^2$$

$$S_0 \sim \int_0^x (\pm i)(t+\pi)^2 dt + A^\pm$$

where we choose the lower limit of integration to be zero for convenience {this just changes the value of the constants $A^\pm$, and then $\exp[A^\pm]$, the overall constant in front is redefined}

$$S_0 \sim \frac{\pm i}{3}(t+\pi)^3 \Big|_0^x + A^\pm$$

$$S_0(x) \sim \frac{\pm i}{3}(x+\pi)^3 - (\pm i)\frac{\pi^3}{3} + A^\pm$$

at $O(\varepsilon)$: $\quad S_1(x) \sim -\frac{1}{4}\ln(x+\pi)^4 + B^\pm$

$$\sim -\ln(x+\pi) + B^\pm$$

So Physical Optics is

$$y(x) \sim \frac{C^+}{(x+\pi)} \exp\left\{\frac{i}{\varepsilon}\left[\frac{(x+\pi)^3}{3} - \frac{\pi^3}{3}\right]\right\}$$

$$+ \frac{C^-}{(x+\pi)} \exp\left\{\frac{-i}{\varepsilon}\left[\frac{(x+\pi)^3}{3} - \frac{\pi^3}{3}\right]\right\}$$

An equivalent basis is

$$y \sim \frac{C_3}{(x+\pi)} \cos\left\{\frac{1}{\varepsilon}\left[\frac{(x+\pi)^3}{3} - \frac{\pi^3}{3}\right]\right\}$$

$$+ \frac{C_4}{(x+\pi)} \sin\left\{\frac{1}{\varepsilon}\left[\frac{(x+\pi)^3}{3} - \frac{\pi^3}{3}\right]\right\}$$

Apply boundary conditions:

$y(0) = 0 \Rightarrow C_3 = 0$

$y(\pi) = 0 \Rightarrow 0 = \frac{C_4}{2\pi} \sin\left\{\frac{1}{\varepsilon}\left[\frac{(2\pi)^3}{3} - \frac{\pi^3}{3}\right]\right\}$

$$0 = \frac{C_4}{2\pi} \sin\left\{\frac{1}{\varepsilon}\frac{7\pi^3}{3}\right\}$$

obviously we do not want $c_1 = 0$ $\Rightarrow$

$$\sin\left\{\frac{1}{\varepsilon}\,\frac{7\pi^3}{3}\right\} = \sin\left\{\sqrt{\lambda_n}\,\frac{7\pi^3}{3}\right\} = 0$$

$$\sqrt{\lambda_n}\,\frac{7\pi^3}{3} = n\pi \qquad n = 1, 2, 3 \ldots$$

$$\text{or} \quad \lambda_n \sim n^2\pi^2 \left(\frac{3}{7\pi^3}\right)^2 \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\begin{aligned} &\text{So} \quad y(x) \sim \frac{c_n}{(x+\pi)} \sin\left\{\sqrt{\lambda_n}\left[\frac{(x+\pi)^3}{3} - \frac{\pi^3}{3}\right]\right\} \\ &\qquad \text{with} \quad \lambda_n \sim n^2\pi^2\left(\frac{3}{7\pi^3}\right)^2 \qquad \lambda_n \to \infty \end{aligned}}$$

We can "normalize" with

$$\int_0^{\pi} (x+\pi)^4\, y_n^2(x)\, dx = 1 \quad \Rightarrow$$

$$c_n^2 = \frac{6}{7\pi^3}$$

Miraculously, the approximation is extremely good & even for $n = 1$!!