Math 322 | 25

SL problem $\quad [p\phi']' + q\phi = -\lambda\sigma\phi \quad a<x<b$

with appropriate boundary conditions (regular, singular, periodic)

leads to the generalized eigenfunction expansion

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x) \qquad a<x<b$$

for piecewise smooth functions $f(x)$.

By orthogonality: $\displaystyle\int_a^b f(x)\phi_n(x)\sigma(x)\,dx = \int_a^b a_n \phi_n^2(x)\sigma(x)\,dx$

$$\Rightarrow \quad a_n = \frac{\int_a^b f(x)\,\phi_n(x)\,\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\,\sigma(x)\,dx}$$

In practice, we usually need to use a finite truncation

$$f(x) \sim \sum_{n=1}^{M} \alpha_n \phi_n(x)$$

and it is not clear that we should choose $\alpha_n = a_n$? Is this the choice that will give is the "best" approximation for $f(x)$ for any given truncation M? Do the "best" values of α_n change with M ?

To define "best", lets first define the error; again there are different measures of the error.

$$E_1 = \max \left| f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right|$$

Notice that $f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)$ is a function of x; we find the largest in absolute value.

$$E_2 = \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x)\, dx$$

the mean-square error turns out to have nice properties

The choice $\alpha_n = a_n$ minimizes the mean-square error for any choice of M.

To see that this might be the case, we would require

For each i

$$\frac{\partial E_2}{\partial \alpha_i} = 0 \qquad i = 1, 2, 3 \ldots, M$$

Now the unknowns are the α_i, and we want a minimum in the space of α's.

This condition is not enough to ensure a min (could be a max or a saddle), but lets check

$$\frac{\partial E_2}{\partial \alpha_i} = \int_a^b 2\left[F(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)\right] \phi_i(x)\,\sigma(x)\,dx = 0$$

For each $i = 1, 2, 3, \ldots, M$

M equations for M unknowns.

$$\int_a^b 2F(x)\,\phi_i(x)\,\sigma(x)\,dx = \int_a^b 2 \sum_{n=1}^{M} \alpha_n \phi_n(x)\,\phi_i(x)\,\sigma(x)\,dx$$

Now use orthogonality

$\Longrightarrow$

$$\int_a^b f(x)\,\phi_i(x)\,\sigma(x)\,dx = \int_a^b \alpha_i\,\phi_i^2(x)\,\sigma(x)\,dx$$

Now it doesn't matter if we call i something else (m, n, etc.)

$$\alpha_i = \frac{\int_a^b f(x)\,\phi_i(x)\,\sigma(x)\,dx}{\int_a^b \phi_i^2(x)\,\sigma(x)\,dx}$$

$$\Longrightarrow \quad \alpha_n = a_n = \frac{\int_a^b f(x)\,\phi_n(x)\,\sigma(x)\,dx}{\int_a^b \phi_n^2(x)\,\sigma(x)\,dx}$$

So the choice $\alpha_n = a_n$ is a good candidate for minimizing the mean square error, but we don't yet have a proof.

Proof using derivatives is harder than proof using complete-the-square

Start again with

$$E_2 = \int_a^b \left[f(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \sigma(x)\, dx$$

$$= \int_a^b \left\{ f^2(x) - \sum_{n=1}^{M} 2f(x)\alpha_n \phi_n(x) + \left[\sum_{n=1}^{M} \alpha_n \phi_n(x) \right]^2 \right\} \sigma(x)\, dx$$

The third term:

$$\int_a^b \sum_{n=1}^{M} \alpha_n \phi_n(x) \sum_{\ell=1}^{M} \alpha_\ell \phi_\ell(x) \quad \sigma(x)\, dx$$

$$= \sum_{n=1}^{M} \sum_{\ell=1}^{M} \int_a^b \alpha_n \alpha_\ell \ \phi_n(x) \phi_\ell(x)\, \sigma(x)\, dx$$

$$= \sum_{n=1}^{M} \int_a^b \alpha_n \alpha_\ell \ \phi_n(x) \phi_\ell(x)\, \sigma(x)\ \delta_{\ell n}\ dx$$

$$= \int_a^b \sum_{n=1}^{M} \alpha_n^2 \phi_n^2(x)\, \sigma(x)\, dx \qquad \Rightarrow$$

$$E_2 = \int_a^b \left\{ f^2 + \sum_{n=1}^{M} \alpha_n^2 \phi_n^2 - \sum_{n=1}^{M} 2\alpha_n f \phi_n \right\} \sigma\, dx$$

with the last 2 terms dependent on α_n

Isolating the last 2 terms:

$$\sum_{n=1}^{M}\left\{\int_a^b \alpha_n^2 \phi_n^2 \sigma\,dx - \int_a^b 2\alpha_n F \phi_n \sigma\,dx\right\} =$$

$$\sum_{n=1}^{M}\left\{\alpha_n^2 \gamma - 2\alpha_n \beta\right\}$$

where $\gamma = \int_a^b \phi_n^2 \sigma\,dx$, $\beta = \int_a^b F\phi_n \sigma\,dx$

$$\sum_{n=1}^{M} \gamma\left[\alpha_n^2 - 2\alpha_n \frac{\beta}{\gamma}\right] =$$

$$\sum_{n=1}^{M} \gamma\left[\alpha_n - \frac{\beta}{\gamma}\right]^2 - \sum_{n=1}^{M} \gamma\frac{\beta^2}{\gamma^2}$$

Go back to the entire E_2

$$E_2 = \underbrace{\int_a^b F^2 \sigma\,dx}_{\text{independent of } \alpha_n} + \sum_{n=1}^{M} \gamma\left[\alpha_n - \frac{\beta}{\gamma}\right]^2 - \underbrace{\sum_{n=1}^{M} \frac{\beta^2}{\gamma}}_{\text{independent of } \alpha_n}$$

and recall that E_2 by its original form

$$E_2 = \int_a^b \left[F(x) - \sum_{n=1}^{M} \alpha_n \phi_n(x)\right]^2 \sigma(x)\,dx \geq 0$$

To minimize E_2 wrt α_n, set middle term $= 0$

$$\Rightarrow \boxed{\alpha_n = \frac{\beta}{\gamma} = \frac{\int_a^b f \phi_n \sigma \, dx}{\int_a^b \phi_n^2 \sigma \, dx}}$$

and the minimal error is

$$E_2 = \int_a^b f^2 \sigma \, dx - \sum_{n=1}^{M} \frac{\beta^2}{\gamma}$$

$$= \int_a^b f^2 \sigma \, dx - \sum_{n=1}^{M} \alpha_n^2 \gamma \qquad \left\{ \alpha_n^2 = \frac{\beta^2}{\gamma^2} \right\}$$

$$\underbrace{\boxed{E_2 = \int_a^b f^2 \sigma \, dx - \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \sigma \, dx}}_{\text{minimal error}} \geq 0$$

what did we do?

* original definition of E_2
* Expanded; used orthogonality
* completed the square
* chose α_n to minimize E_2

Book Example

For Furier sine series with $\sigma(x)=1$, $a=0$, $b=L$

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{L}{2}$$

$$\Rightarrow \quad E_2 = \int_0^L f^2\,dx - \frac{L}{2}\sum_{n=1}^{M}\alpha_n^2$$

Bessel's Inequality Since $E_2 \geq 0$ with $\sigma(x) > 0$, it follows that

$$\int_a^b f^2\sigma\,dx \geq \sum_{n=1}^{M}\alpha_n^2\int_a^b \phi_n^2\sigma\,dx$$

We also know that

$\sum_{n=1}^{M}\alpha_n^2\int_a^b \phi_n^2\sigma\,dx$ increases with M

Since we are adding positive quantities

Therefore the biggest value of the RHS is achieved for $M \to \infty$ and

Parseval's Equality $$\int_a^b f^2\sigma\,dx = \sum_{n=1}^{\infty}\alpha_n^2\int_a^b \phi_n^2\sigma\,dx$$

$\boxed{\text{Parseval's Equality}}$ is important because it relates an integral "energy" of $f(x)$ to the generalized Fourier coefficients {physical space energy to a Fourier space energy}

We know $f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ and

$$\int_a^b f^2(x)\,\sigma(x)\,dx = \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2(x)\,\sigma\,dx$$

$$= \sum_{n=1}^{\infty} a_n^2 \ell^2 = \sum_{n=1}^{\infty} (a_n \ell)^2$$

$$\int_a^b f^2 \sigma\,dx = \sum_{n=1}^{\infty} (a_n \ell)^2$$

length squared of $f(x)$

sum of the squares of the components of $f(x)$ using the orthogonal basis functions