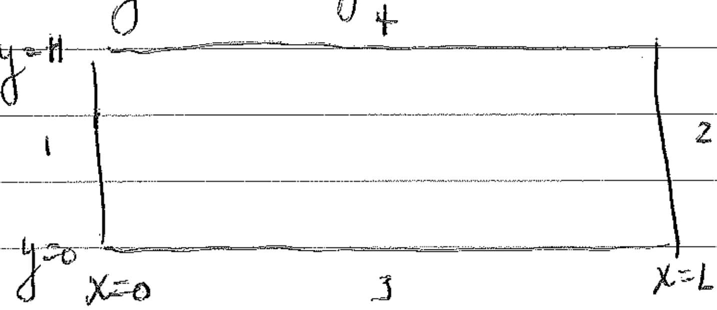
Math 322 | 27

Helmholtz Equation : $\nabla^2 \phi + \lambda \phi = 0$

$$a\phi + b\,\underline{\nabla}\phi \cdot \hat{n} = 0$$

in different geometries.

Vibrating Rectangular Membrane



$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad u = u(x,y) \qquad 0<x<L \quad 0<y<H$$
$$t>0$$

b.c.s $\quad u(0,y,t) = 0 \quad (1) \qquad u(L,y,t) = 0 \quad (2)$

$\quad u(x,0,t) = 0 \quad (3) \qquad u(x,H,t) = 0 \quad (4)$

initial displacement $\quad u(x,y,0) = \alpha(x,y)$

" velocity $\quad \frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$

Separation $u(x,y,t) = \phi(x,y)h(t)$

$$\Rightarrow \quad \frac{1}{c^2 h(t)} \frac{d^2 h(t)}{dt^2} = \frac{1}{\phi(x,y)} \nabla^2 \phi(x,y) = -\lambda$$

Further separation $\phi(x,y) = F(x)\, g(y) \Rightarrow$

$$\boxed{\#1} \quad \frac{1}{F(x)} \frac{d^2 F(x)}{dx^2} = -\mu \quad ; \quad \frac{1}{g(y)} \frac{d^2 g(y)}{dy^2} = -(\lambda - \mu) \quad \boxed{\#2}$$

$F(0) = F(L) = 0$ $\qquad$ $g(0) = g(H) = 0$

$\mu_n = \left(\frac{n\pi}{L}\right)^2$ $\qquad$ $\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2$

$$\Rightarrow \quad \lambda_{nm} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3 \ldots \quad m = 1, 2, 3 \ldots$$

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

$$\boxed{\#3} \quad \frac{d^2 h(t)}{dt^2} = -\lambda_{nm} c^2 h \quad \Rightarrow$$

$$h(t) = A_{nm} \cos\left(c\sqrt{\lambda_{nm}}\, t\right) + B_{nm} \sin\left(c\sqrt{\lambda_{nm}}\, t\right)$$

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \left\{ A_{nm} \cos\left(c\sqrt{\lambda_{nm}}\, t\right) + B_{nm} \sin\left(c\sqrt{\lambda_{nm}}\, t\right) \right\}$$

Now use initial conditions

$$u(x,y,0) = \alpha(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

$$\int_0^L dx \int_0^H dy \quad \alpha(x,y) \sin\frac{n'\pi x}{L} \sin\frac{m'\pi x}{L}$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \int_0^L dx \int_0^H dy \quad \sin\frac{n'\pi x}{L} \sin\frac{n\pi x}{L} \sin\frac{m'\pi y}{H} \sin\frac{m\pi y}{H}$$

$$= A_{nm}\, \delta_{nn'} \frac{L}{2}\, \delta_{mm'} \frac{H}{2}$$

$$\Rightarrow \quad A_{n'm'} = \frac{2}{L}\frac{2}{H} \int_0^L dx \int_0^H dy \quad \alpha(x,y) \sin\frac{n'\pi x}{L} \sin\frac{m'\pi x}{L}$$

Now repeat the process for

$$\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y) \quad \ldots$$

$$\frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \Big\{ -c\sqrt{\lambda_{nm}}\, A_{nm} \sin\left(c\sqrt{\lambda_{nm}}\, t\right)$$

$$+ c\sqrt{\lambda_{nm}}\, B_{nm} \cos\left(c\sqrt{\lambda_{nm}}\, t\right) \Big\}$$

$$\Rightarrow \quad \beta(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, c\sqrt{\lambda_{nm}}\, B_{nm}$$

Use orthogonality $\rightarrow$

$$\sqrt{\lambda_{nm}}\, B_{nm} = \frac{1}{c}\,\frac{2}{H}\,\frac{2}{L} \int_0^L \int_0^H \beta(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}\, dx\, dy$$

Formal solution:

$u(x,y,t) = \ldots$

$A_{nm} = \ldots$

$\sqrt{\lambda_{nm}}\, B_{nm} = \ldots$

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$$

$n = 1, 2, 3 \ldots \qquad m = 1, 2, 3 \ldots$

Now we can represent a 2D function

$$F(x,y) \sim \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

or symbolically

$$F(x,y) = \sum_{\lambda} a_\lambda \Phi_\lambda$$

$$\iint_R F \Phi_{\lambda_i} \, dx\,dy = \sum_{\lambda} \iint_R a_\lambda \Phi_\lambda \Phi_{\lambda_i} \, dx\,dy$$

$$= a_{\lambda_i} \iint_R \Phi_{\lambda_i}^2 \, dx\,dy$$

$$\Rightarrow \quad a_{\lambda_i} = \frac{\iint_R F \Phi_{\lambda_i} \, dx\,dy}{\iint_R \Phi_{\lambda_i}^2 \, dx\,dy}$$

$$a_{nm} = \frac{\int_0^H \int_0^L F(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H} \, dx\,dy}{\int_0^H \int_0^L \sin^2\frac{n\pi x}{L} \sin^2\frac{m\pi y}{H} \, dx\,dy}$$

$$a_{nm} = \frac{2}{L}\frac{2}{H}\int_0^H\int_0^L f(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\,dx\,dy$$

We now also know that for the finite sum

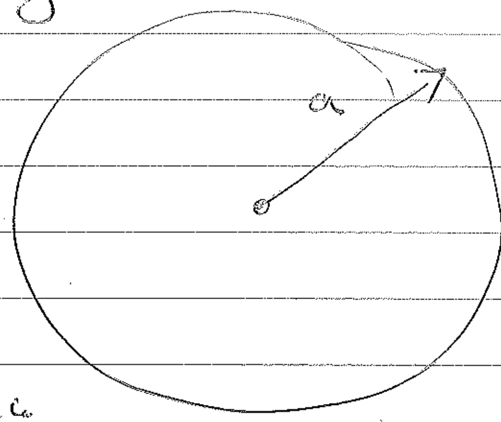
$$f(x,y) \sim \sum_{m=1}^{M}\sum_{n=1}^{N} a_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$$

the formula for a_{nm} given by orthogonality minimizes the mean square error

$$E_2 = \int_0^L\int_0^H \left[f - \sum_{m=1}^{M}\sum_{n=1}^{N} a_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}\right]^2 dx\,dy$$

Now how about a $\boxed{\text{Vibrating Circular Membrane}}$?

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$



$u(r,\theta,t)$

$t > 0 \qquad 0 \le r < a$

$$\left.\begin{array}{l} u(a,\theta,t) = 0 \\ \lim\limits_{r\to 0} |u(r,\theta,t)| < \infty \end{array}\right\} \text{b.c.s}$$

periodicity in θ

initial conditions : $u(r,\theta,0) = \alpha(r,\theta)$

$\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r,\theta)$

Let $u(r,\theta,t) = h(t)\,\phi(r,\theta)$

$$\frac{1}{c^2 h(t)} \frac{d^2 h(t)}{dt^2} = \frac{1}{\phi(r,\theta)} \nabla^2 \phi(r,\theta) = -\lambda$$

So we have Helmholtz again for the spatial dependence : $\nabla^2 \phi(r,\theta) = -\lambda \phi(r,\theta)$

but now in circular polar coordinates

Let $\phi(r,\theta) = f(r)\,g(\theta)$

$$\frac{h''(t)}{c^2 h(t)} = \frac{1}{F(r)g(\theta)} \left[g(\theta) \frac{1}{r} \frac{d}{dr} \left(r \frac{dF(r)}{dr} \right) + F(r) \frac{1}{r^2} \frac{d^2}{d\theta^2} g(\theta) \right] = -\lambda$$

Working on the spatial dependence :

* Multiply through by r^2

$$\frac{r}{F(r)} \frac{d}{dr} \left(r \frac{dF(r)}{dr} \right) + \frac{1}{g(\theta)} \frac{d^2 g(\theta)}{d\theta^2} = -\lambda r^2$$

* Separate r, θ

$$\frac{r}{F(r)} \frac{d}{dr} \left(r \frac{dF(r)}{dr} \right) + \lambda r^2 = -\frac{1}{g(\theta)} \frac{d^2 g(\theta)}{d\theta^2} = \mu$$

Choose + sign !

* $\dfrac{d^2 g(\theta)}{d\theta^2} = -\mu g(\theta)$ with periodic boundary conditions

* $r \dfrac{d}{dr} \left(r \dfrac{dF(r)}{dr} \right) + \lambda r^2 F(r) - \mu F(r) = 0$

is Bessel of order μ

$\boxed{\text{In SL Form}}$

$$\frac{d}{dr}\left(r\frac{dF(r)}{dr}\right) - \frac{\mu F(r)}{r} = -\lambda r F(r)$$

$$p(r) = r, \quad q(r) = -\frac{\mu}{r}, \quad \sigma(r) = r$$

Note that this is not regular:

$p(r) = r$, $\sigma(r) = r$ not strictly positive

but we have boundedness at $r=0$ where
$p(0) = 0$ $q(0) = 0$

AND

$q(r) = -\mu/r = -\frac{m^2}{r}$ is singular at $r=0$

(Note μ is positive so may be renamed $\mu = m^2$)

What does it look like in $\boxed{\text{Standard Form?}}$

$$\frac{dF(r)}{dr} + \frac{r\,d^2F(r)}{dr^2} - \frac{m^2}{r}F(r) = -\lambda r F(r)$$

$$\frac{d^2F(r)}{dr^2} + \frac{1}{r}\frac{dF(r)}{dr} + \left(\lambda - \frac{m^2}{r^2}\right)F(r) = 0$$

which has a regular singular point at $r=0$ since

$$\left.\begin{aligned} r\tilde{p}(r) &= 1 \\ r^2\tilde{q}(r) &= \lambda r^2 - m \end{aligned}\right\} \quad \text{analytic at } r=0$$

With a change of variables $z=\sqrt{\lambda}\, r \Longrightarrow$

$$\frac{d^2F(z)}{dz^2} + \frac{1}{z}\frac{dF(z)}{dz} + \left(1-\frac{m^2}{z^2}\right)F(z) = 0 \quad \text{or}$$

$$z^2F'' + zF' + \left(z^2-m^2\right)F = 0$$

and we can use the method of Frobenius

$$F(z) = z^{\alpha}\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+\alpha}$$

$$\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)\, z^{n+\alpha} + \sum_{n=0}^{\infty} a_n (n+\alpha)\, z^{n+\alpha}$$

$$+ \sum_{n=0}^{\infty} a_n z^{n+\alpha+2} + \sum_{n=0}^{\infty} (-m^2)\, a_n z^{n+\alpha} = 0$$

z^α : $a_0\alpha(\alpha-1) + a_0\alpha - m^2 a_0 = 0$

$a_0\left[\alpha^2 - m^2\right] = 0 \Rightarrow a_0$ arbitrary

$\alpha = \pm m$

So we know that there are 2 linearly independent solutions

$J_m(z) = z^m \sum_{n=0}^{\infty} a_n z^n$ ← bounded at $z=0$

$Y_m(z) = z^{-m} \sum_{n=0}^{\infty} b_n z^n$ ← not bounded at $z=0$

Summarizing

t- dependence $\cos(c\sqrt{\lambda_{mn}}\, t)$ $\sin(c\sqrt{\lambda_{mn}}\, t)$

θ- dependence $\cos m\theta$ $m = 0, 1, 2 \ldots$ $\{\mu = m^2\}$

$\sin m\theta$ $m = 1, 2, 3 \ldots$

r- dependence $J_m(\sqrt{\lambda_{mn}}\, r)$

where $J_m(\sqrt{\lambda_{mn}}\, a) = 0$ determines

the values of $\sqrt{\lambda_{mn}}$

* Note that the circularly symmetric case

$u(r,\theta,0) = \alpha(r)$ (no dependence on θ)

$\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)$ (")

$\Longrightarrow$ $m=0$ and $J_0(\sqrt{\lambda_n}\, a)=0$

For non-symmetric initial conditions we may have $m=0,1,2,\ldots$

* Note that periodicity gives 2 families of eigenfunctions for the same eigenvalue m!

$\cos m\theta$, $\sin m\theta$ $m=1,2,3,\ldots$

* For a special case

$u(r,\theta,0) = \alpha(r,\theta)$

$\frac{\partial u}{\partial t}(r,\theta,0) = 0$

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \cos m\theta \cos(c\sqrt{\lambda_{mn}}\, t)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}\, r) \sin m\theta \cos(c\sqrt{\lambda_{mn}}\, t)$$

$\alpha(r,\theta) = \ldots$

Now A_{mn}, B_{mn} found by orthogonality

$$\int_0^{2\pi} d\theta \int_0^a \phi_\lambda(r,\theta)\, \phi_{\lambda'}(r,\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^a \phi_\lambda^2(r,\theta)\, \delta_{\lambda\lambda'}\, r\, dr\, d\theta$$

weight function in
r is a geometric factor