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Vibrating Membranes in 2D : Non-homogeneous Wave Eqn.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + Q(\underline{x},t) \qquad u = u(\underline{x},t)$$

Let $u(\underline{x},t) = \sum_i A_i(t)\, \phi_i(\underline{x})$

using orthogonality $A_i(t) = \dfrac{\iint u(\underline{x},t)\,\phi_i(\underline{x})\, d\underline{x}}{\iint \phi_i^2(\underline{x})\, d\underline{x}}$

Notes

* This expression is not a soln. for $A_i(t)$ bc. $u(\underline{x},t)$ is the unknown

* The $d\underline{x}$ does indeed account for the weight functions in non-Cartesian coordinates, e.g.

  $r\,dr\,d\theta = d\underline{x}$ in circular polar

$$Q(\underline{x},t) = \sum_i q_i^\circ(t)\,\phi_i(\underline{x}) \quad \Longrightarrow$$

$$q_i^\circ(t) = \frac{\iint Q(\underline{x},t)\,\phi_i(\underline{x})\,d\underline{x}}{\iint \phi_i^2(\underline{x})\,d\underline{x}}$$ known

$\boxed{\text{Method 2}}$ Plug in ; do not differentiate in space term-by-term

$$\sum_i A_i''(t)\,\phi_i(\underline{x}) = c^2 \nabla^2 u(\underline{x},t) + \sum_i q_i^\circ(t)\,\phi_i(\underline{x})$$

Use orthogonality :

$$A_i''(t) = \frac{\iint c^2(\nabla^2 u)\,\phi_i\,d\underline{x}}{\iint \phi_i^2\,d\underline{x}} + q_i^\circ(t)$$

Now integrate by parts twice :

$$A_i^{\circ\prime\prime}(t) = q_i^{\circ}(t) + \frac{\left\{ \iint c^2 u \nabla^2 \phi_i^{\circ}\, d\underline{x} + \oint \phi_i^{\circ} \underline{\nabla} u \cdot \hat{n}\, ds - \oint u \underline{\nabla} \phi^{\circ} \cdot \hat{n}\, ds \right\}}{\iint \phi_i^{\circ\,2}\, d\underline{x}}$$

with boundary terms appearing from integration by parts.

IF $u\big|_{\Omega} = 0$, then

$$A_i^{\circ\prime\prime}(t) = \frac{\iint c^2 u \nabla^2 \phi_i\, d\underline{x}}{\iint \phi_i^2\, d\underline{x}} + q_i^{\circ}(t)$$

$$= \frac{c^2 \iint u(-\lambda_i^{\circ})\, \phi_i^{\circ}\, d\underline{x}}{\iint \phi_i^{\circ\,2}\, d\underline{x}} + q_i(t)$$

$$A_i''(t) = -\lambda_i A_i(t)\, c^2 + q_i(t)$$

Same as Option/Method 1 ...

but note that Option 2 works even if

$u\big|_{\Omega} \neq 0$; keep boundary terms

Lets do this integration by parts carefully
[important going forward]

$$\iint \nabla^2 u \, \phi_i \, d\underline{x} = \iint \phi_i \, \underline{\nabla} \cdot (\underline{\nabla} u) \, d\underline{x}$$

Use the identity

$$\iint \underline{\nabla} \cdot (\phi_i \, \underline{\nabla} u) \, d\underline{x} = \qquad \text{(chain rule)}$$

$$\iint \underline{\nabla} \phi_i \cdot \underline{\nabla} u \, d\underline{x} + \iint \phi_i \, \underline{\nabla} \cdot (\underline{\nabla} u) \, dx$$

$\Rightarrow$

$$\iint \nabla^2 u \, \phi_i \, d\underline{x} = \iint \underline{\nabla} \cdot (\phi_i \underline{\nabla} u) \, d\underline{x} - \iint \underline{\nabla} \phi_i \cdot \underline{\nabla} u \, d\underline{x}$$

Similarly

$$\iint \underline{\nabla} \cdot (u \, \underline{\nabla} \phi_i) \, d\underline{x} = \qquad \text{(chain rule)}$$

$$\iint \underline{\nabla} u \cdot \underline{\nabla} \phi_i \, d\underline{x} + \iint u \, \underline{\nabla} \cdot (\underline{\nabla} \phi_i) \, d\underline{x}$$

$\Longrightarrow$

$$-\iint \underline{\nabla} u \circ \underline{\nabla}\phi_i^{\circ}\, d\underline{x}$$

$$= -\iint \underline{\nabla}\circ(u\,\underline{\nabla}\phi_i^{\circ})\,d\underline{x} + \iint u\,\nabla^2\phi_i^{\circ}\,dx$$

Combine the two:

$$\iint \nabla^2 u\ \phi_i^{\circ}\,d\underline{x} = \iint \underline{\nabla}\circ(\phi_i^{\circ}\,\underline{\nabla} u)\,d\underline{x}$$

$$-\iint \underline{\nabla}\circ(u\,\underline{\nabla}\phi_i^{\circ})\,d\underline{x} + \iint u\,\nabla^2\phi_i^{\circ}\,d\underline{x}$$

$$= \oint \phi_i^{\circ}\,\underline{\nabla} u\circ\hat{n}\,ds - \oint u\,\underline{\nabla}\phi_i^{\circ}\circ\hat{n}\,ds$$

$$+\iint u\,\nabla^2\phi_i^{\circ}\,d\underline{x}$$

by divergence thm

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$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u + Q(\underline{x}, t) \qquad u = u(\underline{x}, t)$$

$$u\big|_{\Omega} = 0 \qquad u(\underline{x}, 0) = \alpha(\underline{x}) \qquad \frac{\partial u}{\partial t}(\underline{x}, 0) = \beta(\underline{x})$$

In <u>any</u> geometry:

$$u(\underline{x}, t) = \sum_i A_i(t)\, \phi_i(\underline{x}) \qquad Q(\underline{x}, t) = \sum_i q_i(t)\, \phi_i(\underline{x})$$

$$\frac{d^2 A_i(t)}{dt^2} + c^2 \lambda_i A_i(t) = q_i(t)$$

2^{nd}-order, linear, constant coefficients, non-homogeneous

$$A_i(t) = A_i^h(t) + A_i^P(t)$$

$A_i^h(t)$ is the homogeneous soln.

$A_i^P(t)$ is a particular solution

Since $c^2, \lambda_i > 0$

$$A_i^h(t) = a_i \cos(c\sqrt{\lambda_i}\, t) + b_i \sin(c\sqrt{\lambda_i}\, t)$$

and we need to use Variation of Parameters or Undetermined Coefficients to find $A_i^P(t)$

Lets Consider Special Case

$$Q(\underline{x},t) = \bar{Q}(\underline{x}) \cos\omega t = \sum_i q_i(t)\,\phi_i(\underline{x})$$

with

$$q_i(t) = \frac{\iint \bar{Q}(\underline{x})\,\phi_i(\underline{x}) \cos\omega t \, d\underline{x}}{\iint \phi_i^2(\underline{x})\, d\underline{x}}$$

$$= \gamma_i \cos\omega t$$

$$A_i''(t) + c^2 \lambda_i A_i(t) = \gamma_i \cos\omega t$$

As we learned in ODEs, 2 subcases

① $\omega \neq c\sqrt{\lambda_i}$

② $\omega = c\sqrt{\lambda_i}$ resonance

(1) $\omega \neq c\sqrt{\lambda_i}$

$$A_i^p(t) = B_i \cos\omega t + D_i \sin\omega t$$

Plug in to find :

$$B_i = \frac{\gamma_i}{c^2\lambda_i - \omega^2} \qquad D_i = 0$$

So the general solution is

$$A_i(t) = a_i \cos(c\sqrt{\lambda_i}\, t) + b_i \sin(c\sqrt{\lambda_i}\, t) + \frac{\gamma_i}{c^2\lambda_i - \omega^2} \cos\omega t$$

Now use

$$u(\underline{x},t) = \sum_i \left[a_i \cos(c\sqrt{\lambda_i}\, t) + b_i \sin(c\sqrt{\lambda_i}\, t) + \frac{\gamma_i}{c^2\lambda_i - \omega^2} \cos\omega t \right] \phi_i(\underline{x})$$

$$\frac{\partial u}{\partial t}(\underline{x},t) = \ldots$$

$$u(\underline{x},0) = \alpha(\underline{x}) = \sum_i \left[a_i^\circ + \frac{\gamma_i^\circ}{c^2\lambda_i^\circ - \omega^2} \right] \phi_i^\circ(\underline{x})$$

$$\frac{\partial u}{\partial t}(\underline{x},0) = \beta(\underline{x}) = \cdots$$

Use orthogonality to find a_i°, b_i°

(2) $\omega = c\sqrt{\lambda_i^\circ}$

$A_i^P(t) \neq B_i^\circ \cos\omega t + D_i^\circ \sin\omega t$!! Need

$$A_i^P(t) = B_i^\circ t\cos\omega t + D_i^\circ t\sin\omega t$$

Plug in to find

$$A_i^P(t) = \frac{\gamma_i^\circ}{2\omega} t\sin\omega t \qquad \left\{ \begin{array}{l} B_i^\circ = 0 \\ D_i^\circ = \dfrac{\gamma_i^\circ}{2\omega} \end{array} \right\}$$

Now

$$u(\underline{x},t) = \sum_i \Big[a_i^\circ \cos(c\sqrt{\lambda_i^\circ}\, t) + b_i^\circ \sin(c\sqrt{\lambda_i^\circ}\, t) + \frac{\gamma_i^\circ}{2\omega} t\sin\omega t \Big] \phi_i^\circ(\underline{x})$$

grows without bound

Again find a_i, b_i using the initial conditions

Note that damping "regularizes" the PDE such that exact resonance cannot occur

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u - \beta \frac{\partial u}{\partial t} + Q(\underline{x}, t)$$

$$u\big|_{\Omega} = 0 \quad , \quad u(\underline{x}, 0) = \alpha(\underline{x}) \quad , \quad \frac{\partial u}{\partial t}(\underline{x}, 0) = \beta(\underline{x})$$

This is the exam problem with a non-homogeneous term

$\boxed{\text{Next}}$

* Intro to Green's Functions

* Method of Characteristics for Wave Eqns.

* ~~Intro to Nonlinear PDEs~~
  ~~quasi-linear PDEs and shocks~~