Math 322 | 36

① What is the relation between the Green's function and the eigenfunction solution?

Which approach should we choose?

② What is the infinite-space Green's function for Laplace's Equation / Poisson's Eqn. in 3D?

Method of Images ...

③ Green's function approach for other linear operators? [Different operators from $\underline{\nabla} \cdot \underline{\nabla}$]

Revisit $\nabla^2 u(\underline{x}) = f(\underline{x})$ with non-homogeneous boundary conditions / 2D

The infinite space Green's function in 2D given by $\nabla^2 G(\underline{x}; \underline{x}_0) = \delta(\underline{x} - \underline{x}_0)$

is $G(\underline{x}; \underline{x}_0) = \frac{1}{2\pi} \ln\left\{\left[(x-x_0)^2 + (y-y_0)^2\right]^{1/2}\right\}$

$= \frac{1}{2\pi} \ln r$

Semi-infinite domain problem: 2D

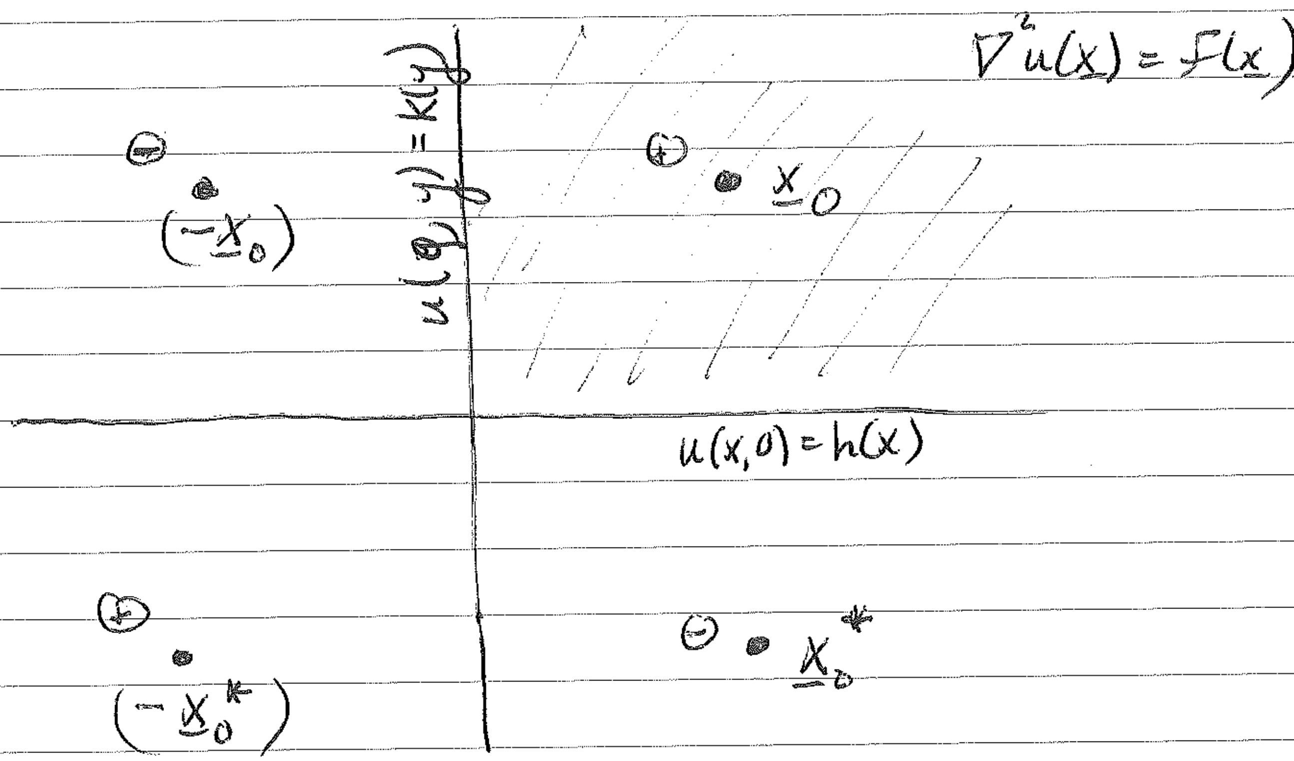
$$\nabla^2 u(\underline{x}) = f(\underline{x}) \qquad 0 < y < \infty \qquad -\infty < x < \infty$$
$$u(x,0) = h(x)$$

Method of Images $\Rightarrow$

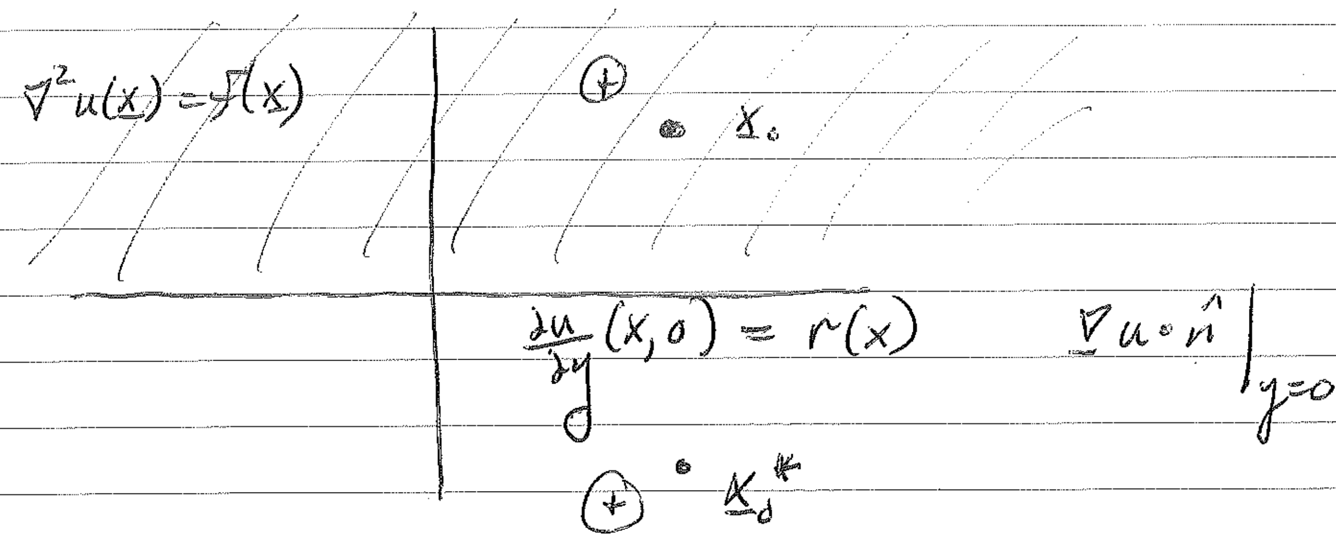
$$G(\underline{x};\underline{x}_0) = \frac{1}{2\pi}\ln|\underline{x}-\underline{x}_0| - \frac{1}{2\pi}\ln|\underline{x}-\underline{x}_0^*|$$

$\underline{x} = (x,y)$ $\quad \underline{x}_0 = (x_0, y_0)$ upper half plane $\qquad \underline{x}_0^* = (x_0, -y_0)$ lower half plane

Quarter plane with Dirichlet b.c.s.



Upper half plane with Neumann b.c.s



What is the relation with eigenfunction expansions?

Recall our 3 methods: (non-homogeneous problems)

(i) Reference Function

$$u(\underline{x},t) = r(\underline{x},t) + \sum_i a_i(t)\,\phi_i(\underline{x})$$

(ii) Eigenfunction expansion / integration by parts

$$u(\underline{x},t) = \sum_i a_i(t)\,\phi_i(\underline{x})$$

(iii) Green's function (infinite space) with

Method of Images.

Lets relate them for a particular problem:

$\nabla^2 u(\underline{x}) = F(\underline{x}) \quad 0<x<L, \; 0<y<H$ with Dirichlet boundary conditions

The abstract eigenfunction expansion

* Solve $-\nabla^2 \phi(\underline{x}) = \lambda\, \phi(\underline{x})$ to find $\lambda_1, \lambda_2, \dots \lambda_n, \dots$

  $\phi_{\lambda_i}(\underline{x})$ or $\phi_i(\underline{x})$ for short

* By SL theory, we know that the eigenfunctions are complete and orthogonal

$$\int \phi_i(\underline{x})\, \phi_j(\underline{x})\, d\underline{x} = \delta_{ij}\, I$$

* For $\nabla^2 u(\underline{x}) = F(\underline{x})$ use the expansions

$$u(\underline{x}) = \sum_i a_i\, \phi_i(\underline{x}) \qquad F(\underline{x}) = \sum_i b_i\, \phi_i(\underline{x})$$

$$a_i = \frac{\int u\, \phi_i\, d\underline{x}}{I}, \quad b_i = \underbrace{\frac{\int F\, \phi_i\, d\underline{x}}{I}}_{\text{known}}$$

* Plug in $\nabla^2 \sum_i a_i \phi_i(\underline{x}) = \sum_i b_i \phi_i(\underline{x})$

** If homogeneous b.c.s on $u(\underline{x})$, then differentiate term by term {no boundary terms generated}

** If non-homogeneous b.c.s on $u(\underline{x})$, then integrate by parts

* $\sum_i (-\lambda_i) a_i \phi_i(\underline{x}) = \sum_i b_i \phi_i(\underline{x}) + \boxed{\text{boundary terms}}$

* Use orthogonality

$$\int \phi_j \sum_i (-\lambda_i) a_i \phi_i \, d\underline{x} = \int \phi_j \sum_i b_i \phi_i \, d\underline{x}$$

$$+ \int \phi_j \boxed{} \, d\underline{x}$$

$$-\lambda_j a_j I = b_j I + \triangle$$

$$a_i = -\frac{1}{\lambda_i} b_i + \frac{1}{I} \triangle$$

For the rectangular domain:

$$\lambda_i \Rightarrow \lambda_{mn} = \frac{m^2\pi^2}{H^2} + \frac{n^2\pi^2}{L^2}$$

$$\phi_i \Rightarrow \phi_{nm} = \sin\frac{n\pi x}{L}\,\sin\frac{m\pi y}{H} \qquad \begin{matrix} n=1,2,3\ldots \\ m=1,3,3\ldots \end{matrix}$$

For the special case $\Delta = 0$ { $u(\underline{x})$ had homogeneous boundary conditions } { generalizable }

$$a_i = -\frac{1}{\lambda_i}\, b_i \quad \Rightarrow$$

$$u(\underline{x}) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-\frac{1}{\lambda_{nm}}\right) b_{nm}\,\sin\frac{n\pi x}{L}\,\sin\frac{m\pi y}{L}$$

$$b_{nm} = \frac{\int_0^L dx_0 \int_0^H dy_0\; F(\underline{x}_0)\,\sin\frac{n\pi x_0}{L}\,\sin\frac{m\pi y_0}{H}}{\int_0^L dx_0 \int_0^H dy_0\; \sin^2\frac{n\pi x_0}{L}\,\sin^2\frac{m\pi y_0}{H}}$$

$$= \frac{4}{LH}\int_0^L dx_0 \int_0^H dy_0\; F(\underline{x}_0)\,\sin\frac{n\pi x_0}{L}\,\sin\frac{m\pi y_0}{H}$$

Finally $\quad u(\underline{x}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty}$

$$\left\{ \frac{\frac{-4}{LH} \int_0^L dx_0 \int_0^H dy_0 \, F(\underline{x}_0) \sin\frac{n\pi x_0}{L} \sin\frac{m\pi y_0}{H}}{\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)} \right\} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{H}$$

or

$$u(\underline{x}) = \int_0^L dx_0 \int_0^H dy_0 \, F(\underline{x}_0)$$

$$\left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{-4}{LH}\right) \frac{\sin\frac{n\pi x_0}{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi y_0}{H} \sin\frac{m\pi y}{H}}{\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)} \right\}$$

$$= \int_{A_0} F(\underline{x}_0) G(\underline{x}_0; \underline{x}) \, dA_0 + \cancel{u_h(\underline{x})} \xrightarrow{} 0$$ in this case

$G(\underline{x}; \underline{x}_0) = G(\underline{x}_0; \underline{x})$ symmetric

$$= \frac{-4}{LH} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin\frac{n\pi x}{L} \sin\frac{n\pi x_0}{L} \sin\frac{m\pi y}{H} \sin\frac{m\pi y_0}{H}}{\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}\right)}$$

an infinite number of image points ?