Math 322 | 38

Method of Characteristics for the Wave Equation; a strategy to transform a PDE to equivalent ODEs

1^{st} consider the infinite domain $-\infty < x < \infty$

$u = u(x,t)$ in 1D

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{same as}$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

check: $\frac{\partial^2 u}{\partial t^2} + \cancel{c\frac{\partial^2 u}{\partial x \partial t}} - \cancel{c\frac{\partial^2 u}{\partial t \partial x}} - c^2\frac{\partial^2 u}{\partial x^2} = 0$ ✓

also same as $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$

Defining $w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$; $v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$

$\Rightarrow$ 2 1^{st}-order wave equations

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad ; \quad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

$w = w(x,t)$ $\qquad$ $v = v(x,t)$

1st understand the 1st-order wave equation; then return to the 2nd-order case.

Let $x = x(t)$ be a "moving observer" [in fluids, let $x = x(t)$ "follow a fluid particle"]

Then $w = w(x(t), t)$ satisfies

$$\frac{d}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial t} \quad \text{by chain rule}$$

If the observer moves at (constant) speed $\frac{dx(t)}{dt} = c$, then the 1D wave eqn. is equivalent to

$$\boxed{\frac{dw}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = c \qquad \text{where } w = w(x(t), t)\,; \quad x = x(t)}$$

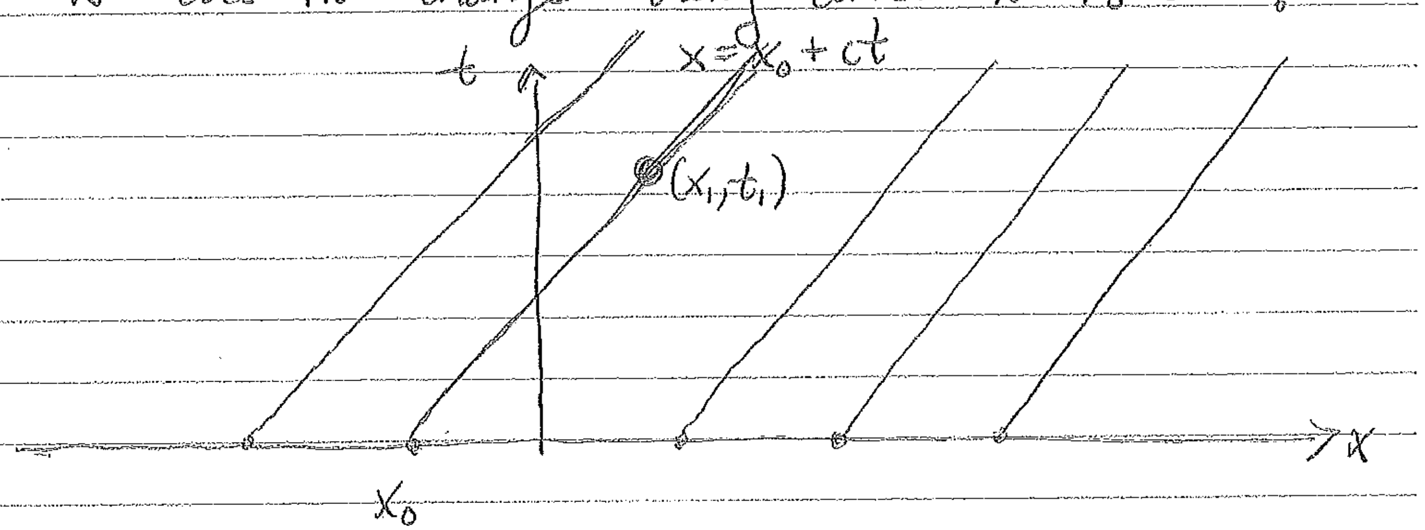
Thus we have converted the PDE into 2 ODEs by introducing a moving observer.

We are using the "method of characteristics" and the curves given by $\frac{dx(t)}{dt} = c$ are called the "characteristic curves".

In this case $x = x_0 + ct$ are lines with slope c originating at x_0.

$\frac{dw}{dt} = 0$ along $x = x_0 + ct$ means that

w does not change along curves $x = x_0 + ct$:



In the original problem we will be given an initial condition, say $w(x,0) = P(x)$

so we need to solve

$$\frac{dw(x(t),t)}{dt} = 0 \quad \text{along} \quad \frac{dx(t)}{dt} = 0 \quad \text{with}$$

$$w(x(0),0) = P(x(0))$$

Since w does not change in time along characteristics $\Rightarrow$

$w(x,t) = P(x_0)$ on $x = x_0 + ct$, same as

$w(x,t) = P(x_0)$ on $x_0 = x - ct$ $\Rightarrow$

$$\boxed{w(x,t) = P(x-ct)}$$

This is a long-winded way of saying

$w(x,t) = P(x-ct)$ satisfies

$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ which we already knew!

Check: Let $\xi = x - ct$, $w(x,t) = P(\xi)$

$$\frac{\partial w}{\partial t} = -cP'(\xi) \; ; \; c\frac{\partial w}{\partial x} = cP'(\xi) \quad \checkmark$$

$\boxed{\text{Example 1}}$ $\quad \dfrac{\partial w}{\partial t} + 2\dfrac{\partial w}{\partial x} = 0$

$$w(x,0) = P(x) = \begin{cases} 0 & x < 0 \\ 4x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

w is not changing in time along $x = x_0 + 2t$

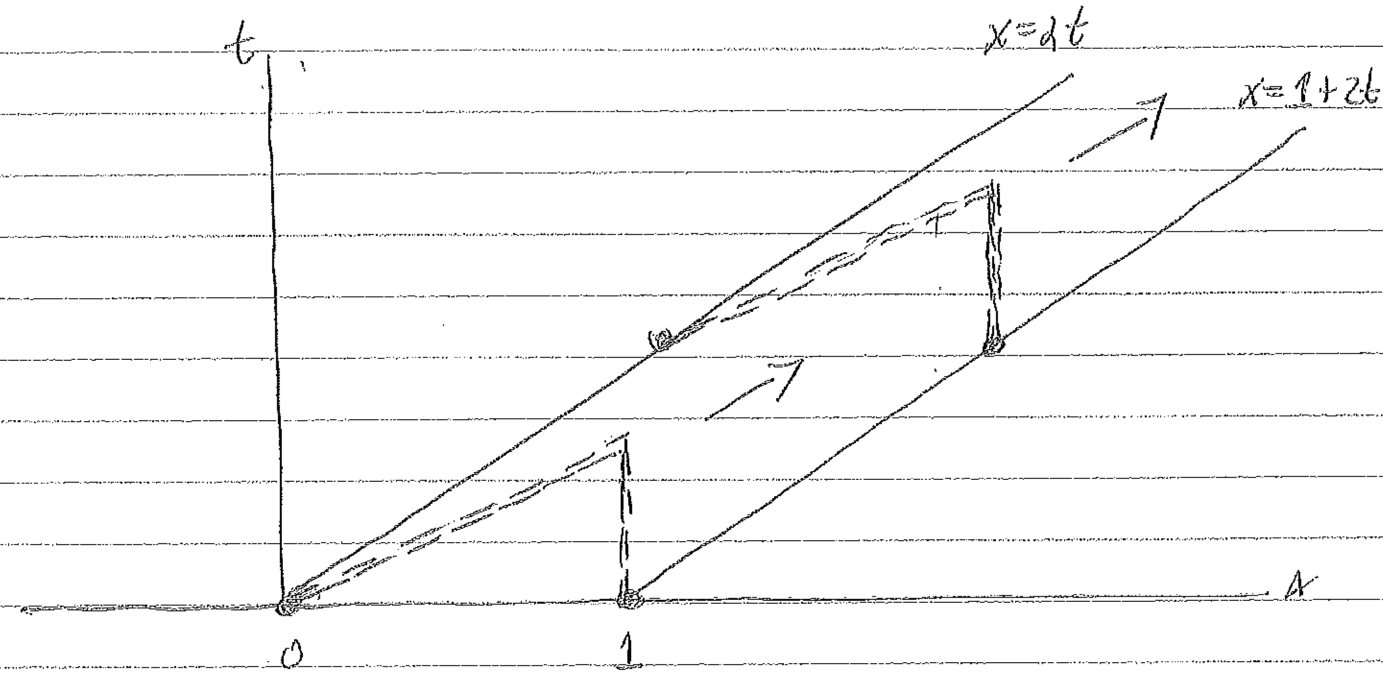
$w(x,t) = P(x_0)$ on $x_0 = x - 2t$

$$= \begin{cases} 0 & x_0 < 0 \\ 4x_0 & 0 \le x_0 \le 1 \\ 0 & x_0 > 1 \end{cases}$$

$$= \begin{cases} 0 & x - 2t < 0 \\ 4(x-2t) & \cancel{0 \le x - 2t \le 0} \\ 0 & x - 2t > 1 \end{cases}$$

or also written

$$W(x,t) = \begin{cases} 0 & x < 2t \\ 4(x-2t) & 0 \le x-2t \le 1 \\ 0 & x > 1+2t \end{cases}$$



imagine anothe axis out of the page

a wave of fixed shape moving to the right.

max amplitude of the wave is 4