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$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad -\infty < x < \infty \qquad t > 0$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

* $F(x-ct)$ is a wave of fixed shape moving right

$G(x+ct)$ " " " " left

The sum changes shape!

* 2 sets of characteristic curves

$$x + ct = \beta \qquad x - ct = \alpha$$

* How do we find F, G?

$u(x,0) = f(x)$: $\qquad f(x) = F(x) + G(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$: $\qquad g(x) = -cF'(x) + cG'(x)$

2 equations for 2 unknowns

f, g known $\qquad F, G$ unknown

(1) $c\mathcal{F}' = F' + cG'$

(2) $g = -cF' + cG'$

Solve for F', G' in terms of $\mathcal{F}$, g; then integrate $\Rightarrow$

$$F(x) = \frac{\mathcal{F}(x)}{2} - \frac{1}{2c}\int_0^x g(s)\,ds - A$$

$$G(x) = \frac{\mathcal{F}(x)}{2} - \frac{1}{2c}\int_0^x g(s)\,ds + A$$

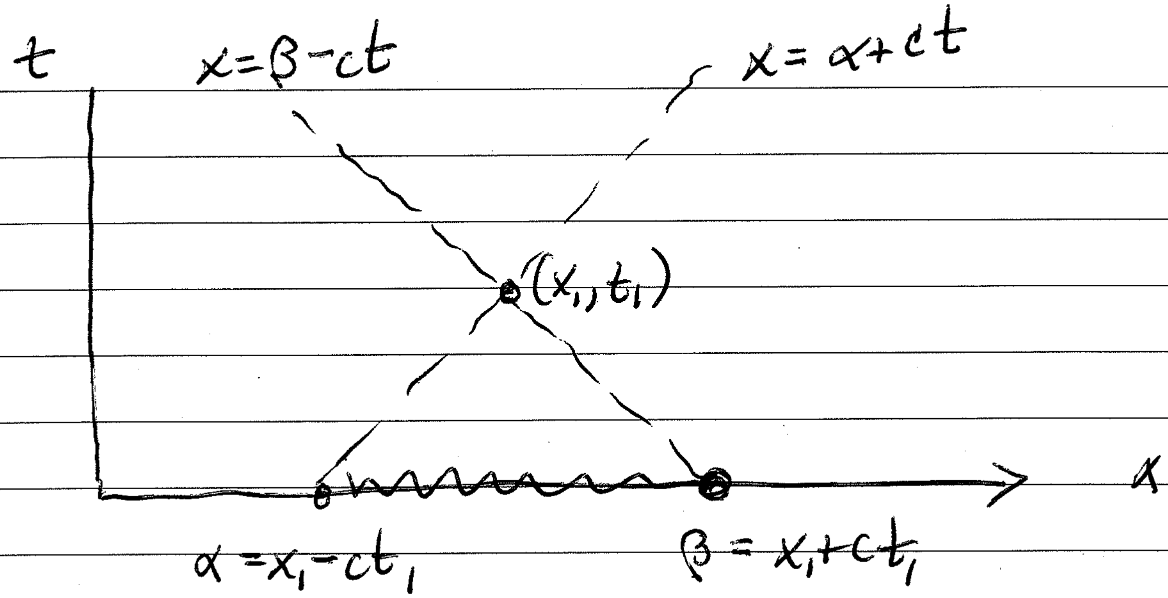
Then since $u(x,t) = F(x-ct) + G(x+ct)$

$$= \frac{1}{2}\mathcal{F}(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds - A$$

$$+ \frac{1}{2}\mathcal{F}(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + A$$

d'Alembert's Solution

$$u(x,t) = \frac{\mathcal{F}(x+ct) + \mathcal{F}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$



To find the solution at (x_1, t_1) we need:

* initial position dat at

$$\alpha = x_1 - ct_1 \qquad \beta = x_1 + ct_1$$

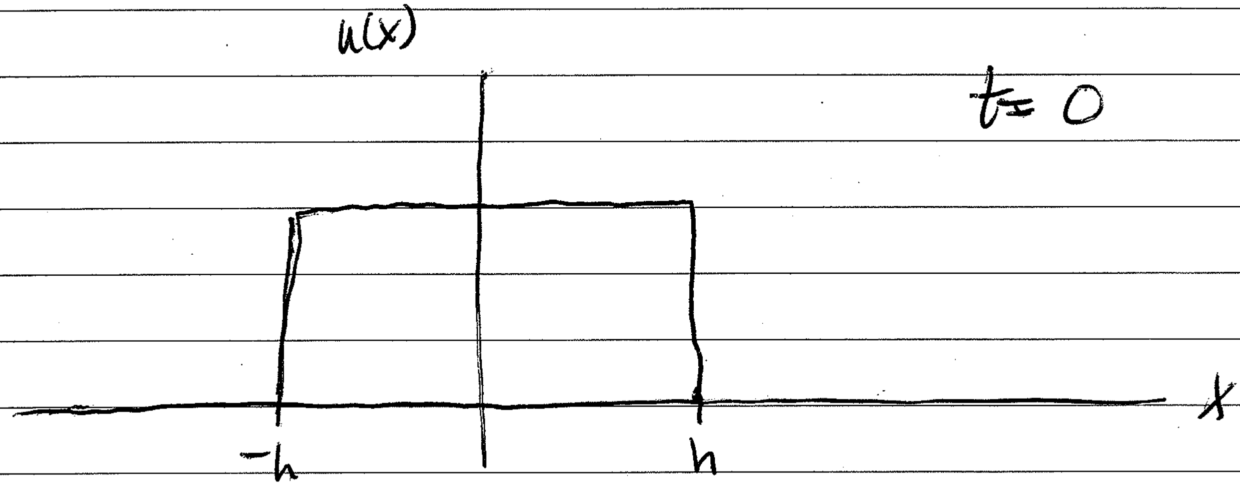
* initial velocity data at

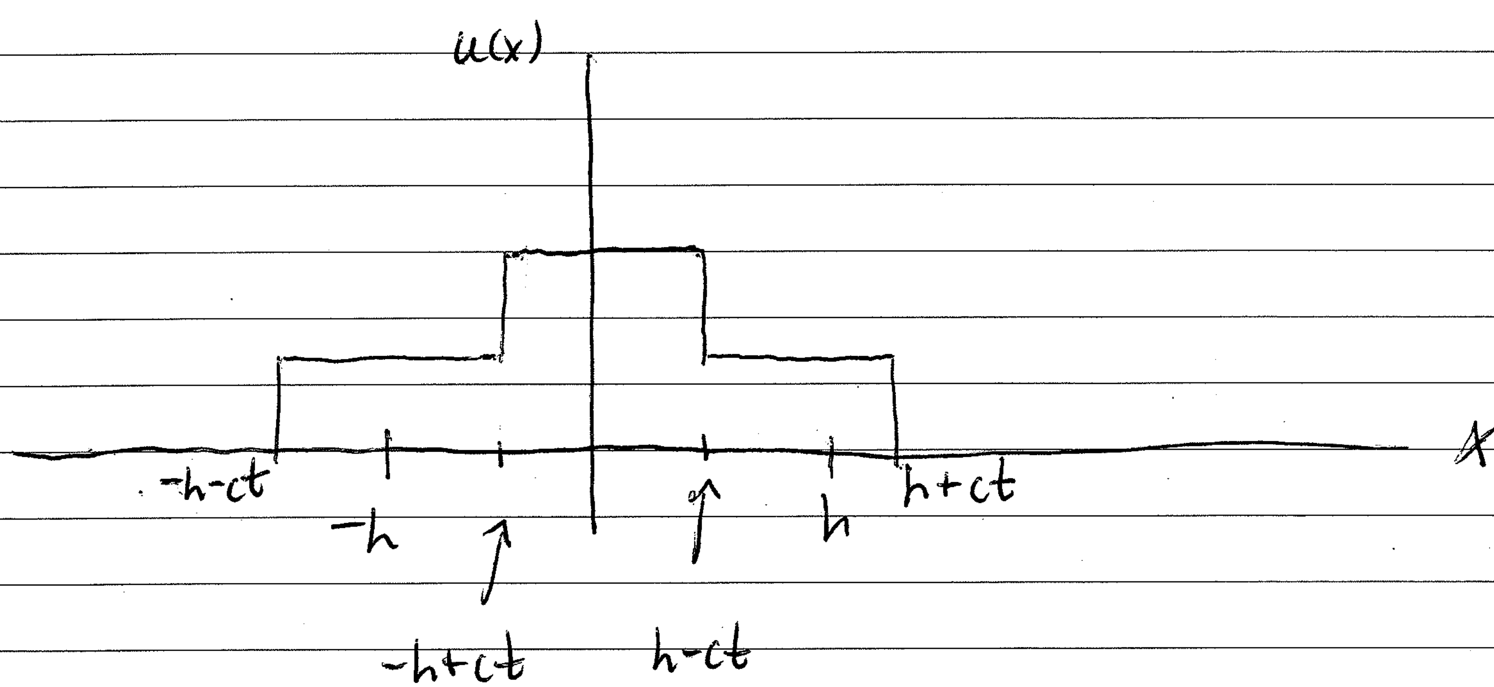
$$x_1 - ct_1 \le x \le x_1 + ct_1$$

Initial data $u(x,0) = f(x)$ $\quad \frac{\partial u}{\partial t}(x,0) = 0$

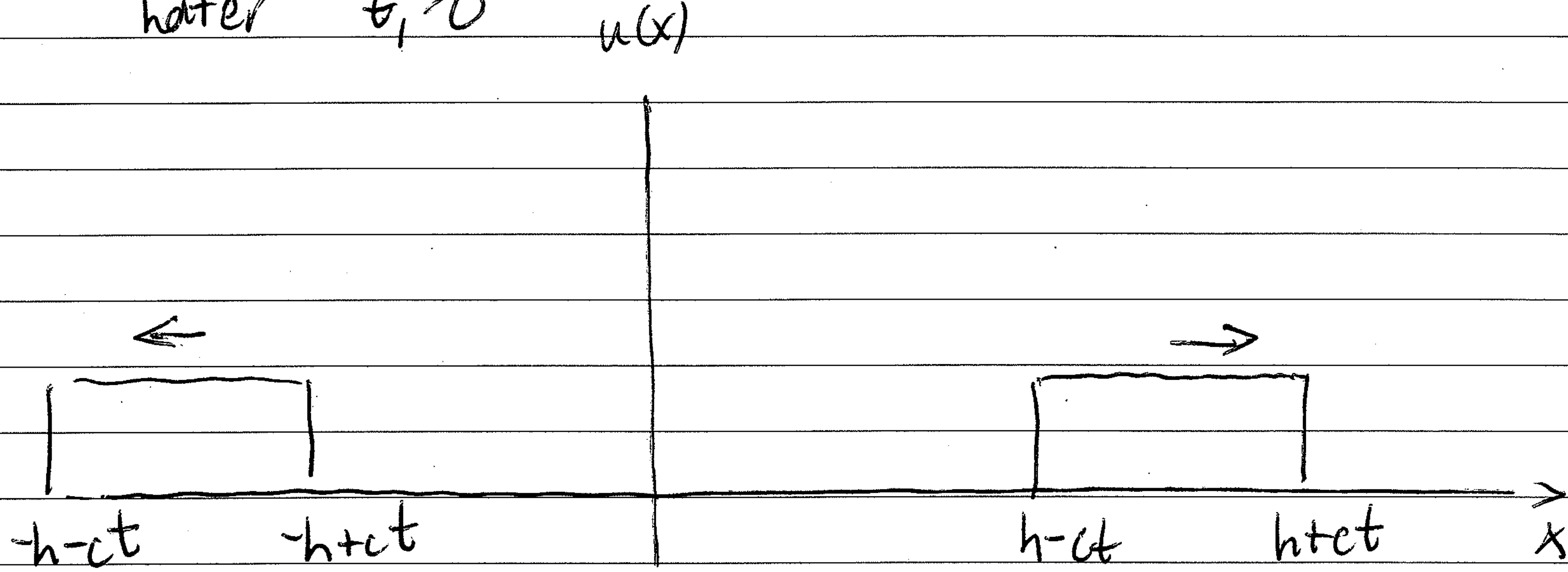
$$f(x) = \begin{cases} 1 & |x| \le h \\ 0 & |x| > h \end{cases}$$

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

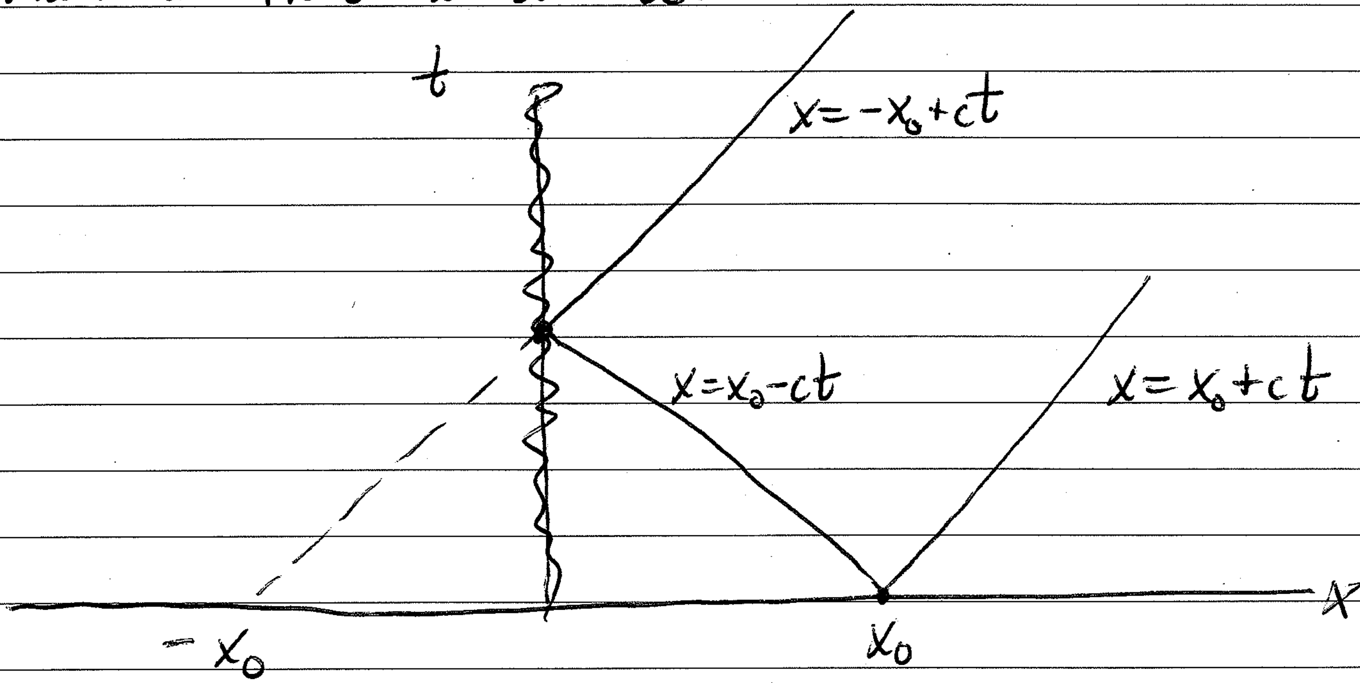




Later $t_1 > 0$



What if there is a wall?



Semi-infinite domain problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x < \infty \qquad t > 0$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad \boxed{u(0,t) = h(t)}$$

new information!

The equation is still satisfied by

$$u(x,t) = F(x-ct) + G(x+ct)$$

3 algebraic equations

$u(x,0) = f(x)$: $\quad f(x) = F(x) + G(x)$

$\frac{\partial u}{\partial t}(x,0) = g(x)$: $\quad g(x) = -cF'(x) + cG'(x)$

$u(0,t) = h(t)$: $\quad h(t) = F(-ct) + G(ct)$

Solving gives

$$u(x,t) = F(x-ct) + G(x+ct)$$

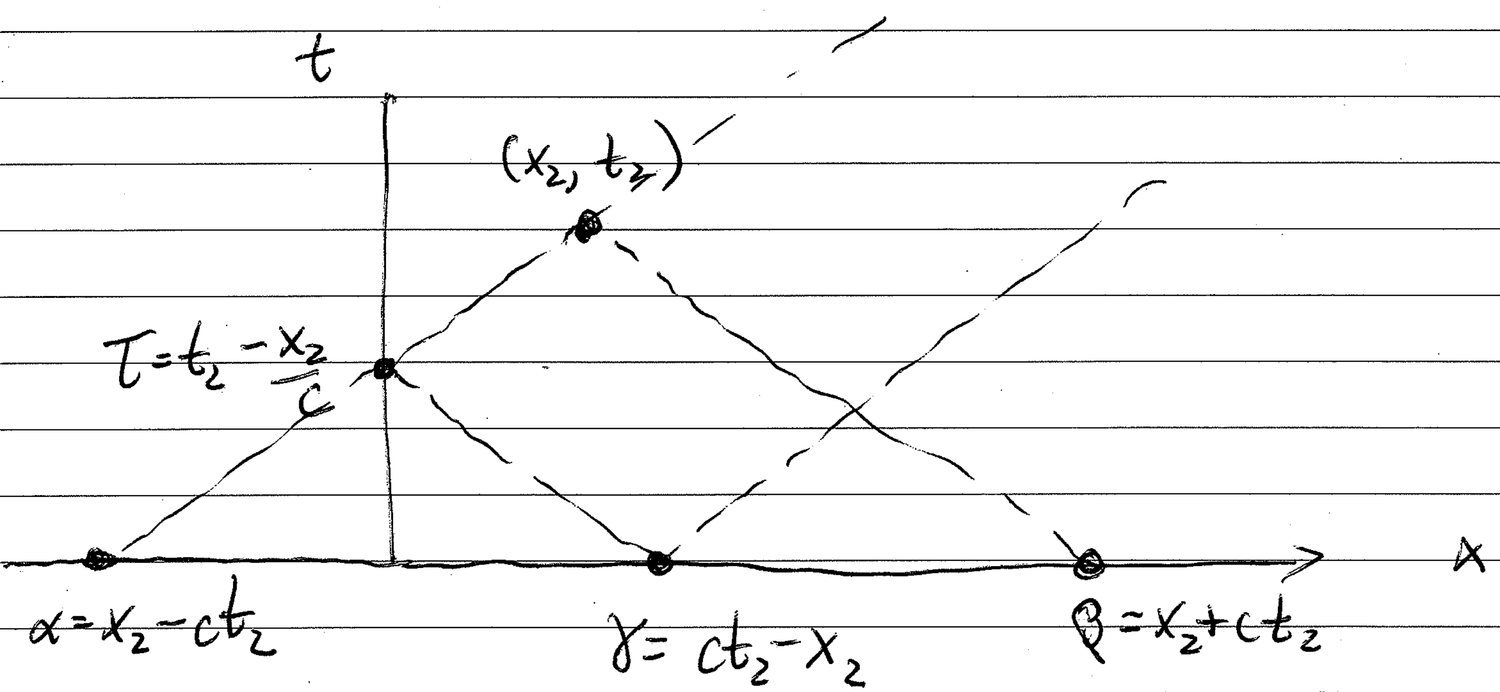
$$= \begin{cases} \dfrac{f(x+ct)+f(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds & x \geq ct \\ h(t - x/c) + \left[\dfrac{f(x+ct) - f(ct-x)}{2}\right] + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct} g(s)\,ds & x < ct \end{cases}$$

* We need initial data only for

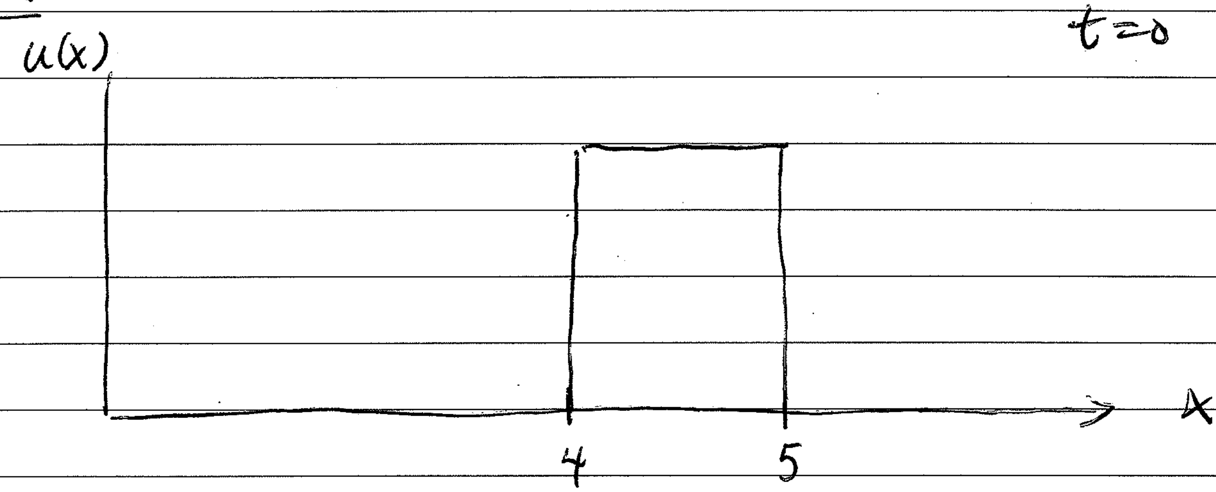
$x \geq ct$ $\{ ct < x$ short times $\}$

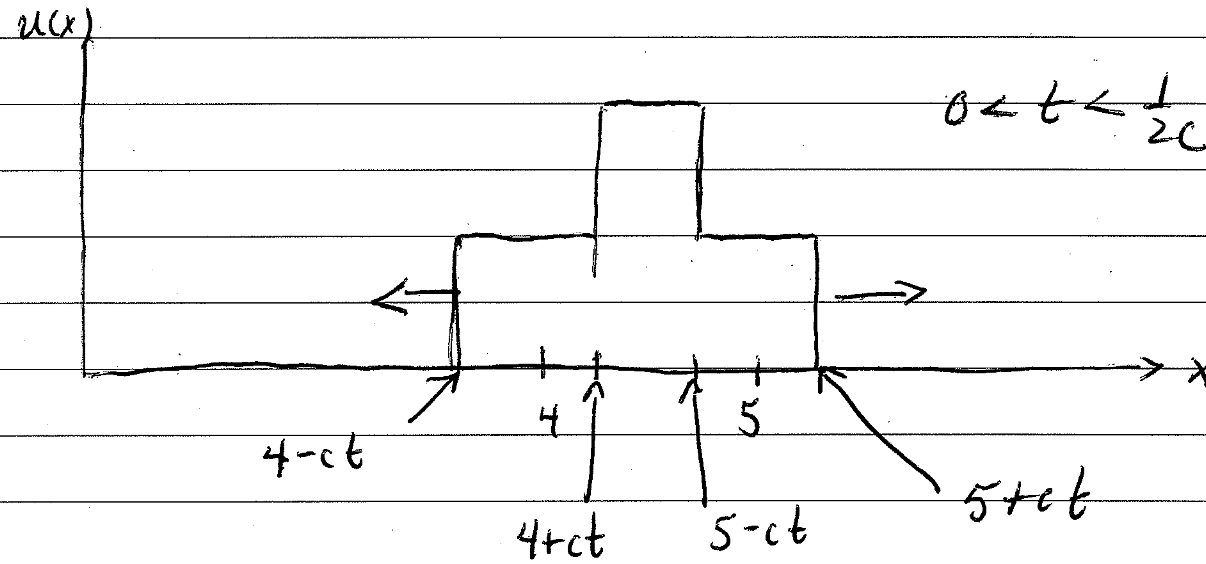
* Need initial data and boundary dat

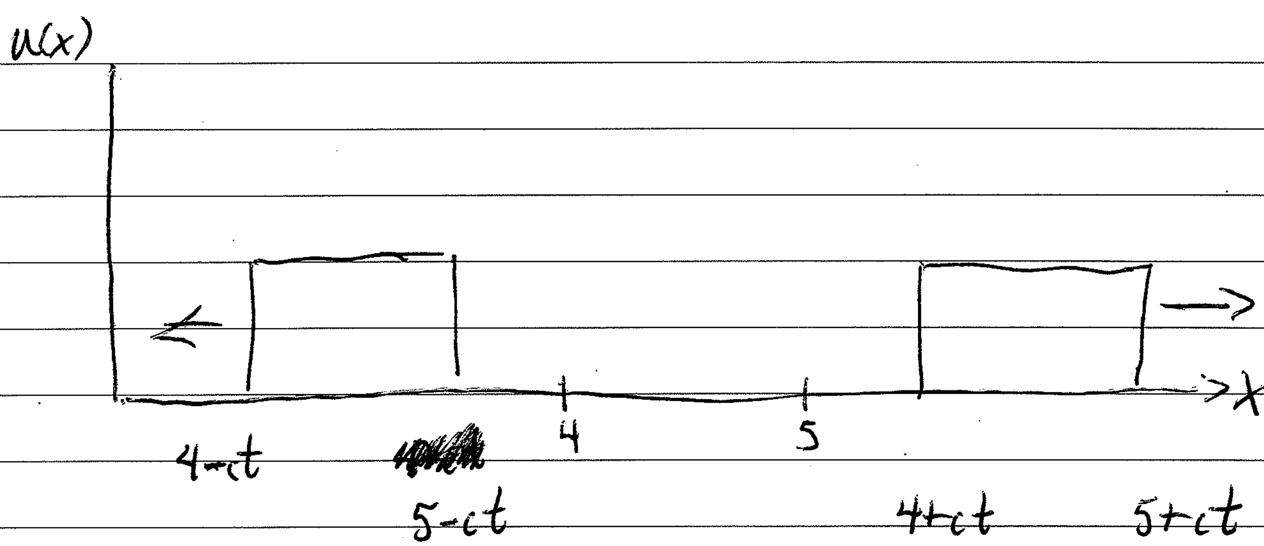
for $x < ct$ $\{ ct > x$ longer times $\}$

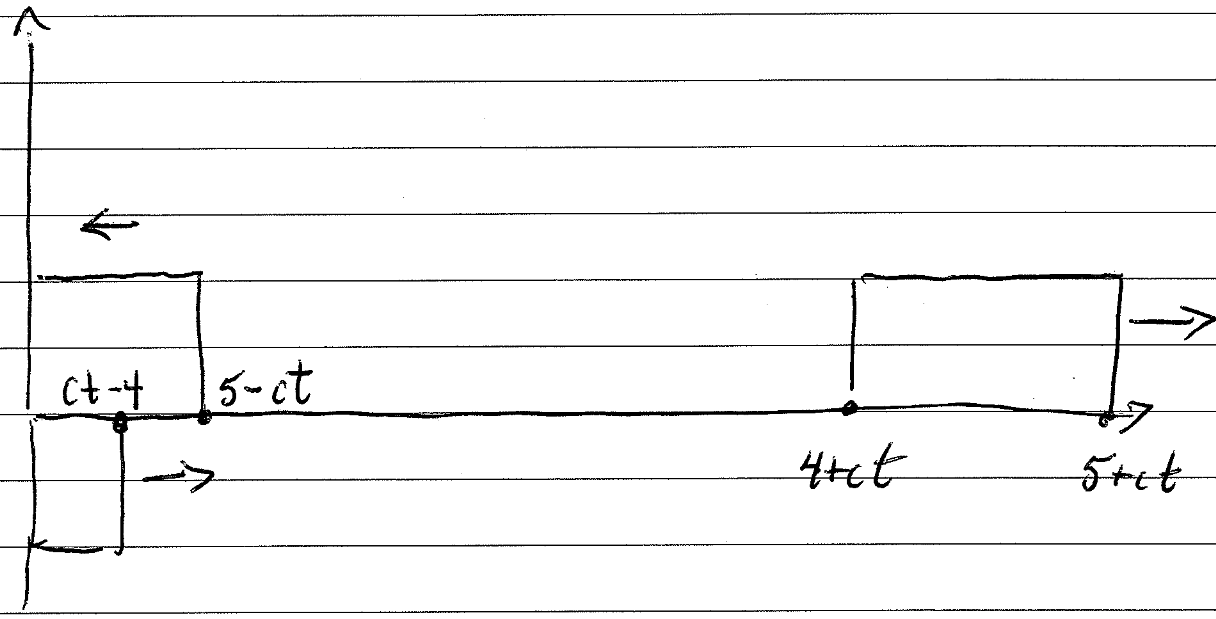


Pictures

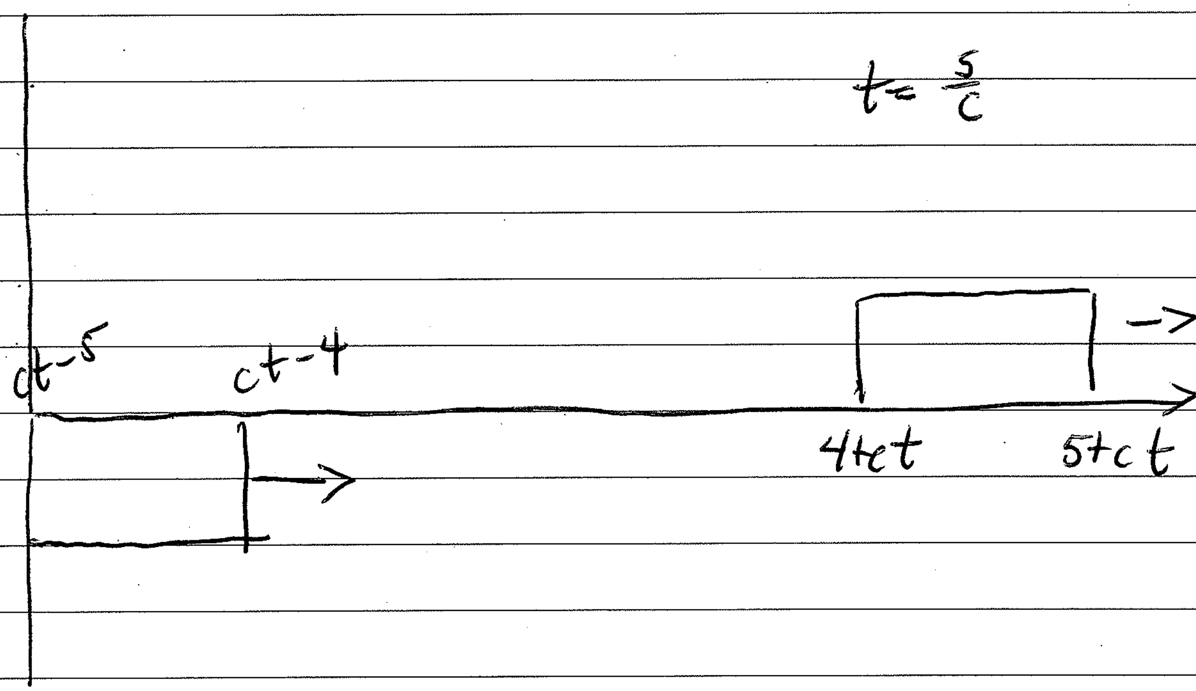




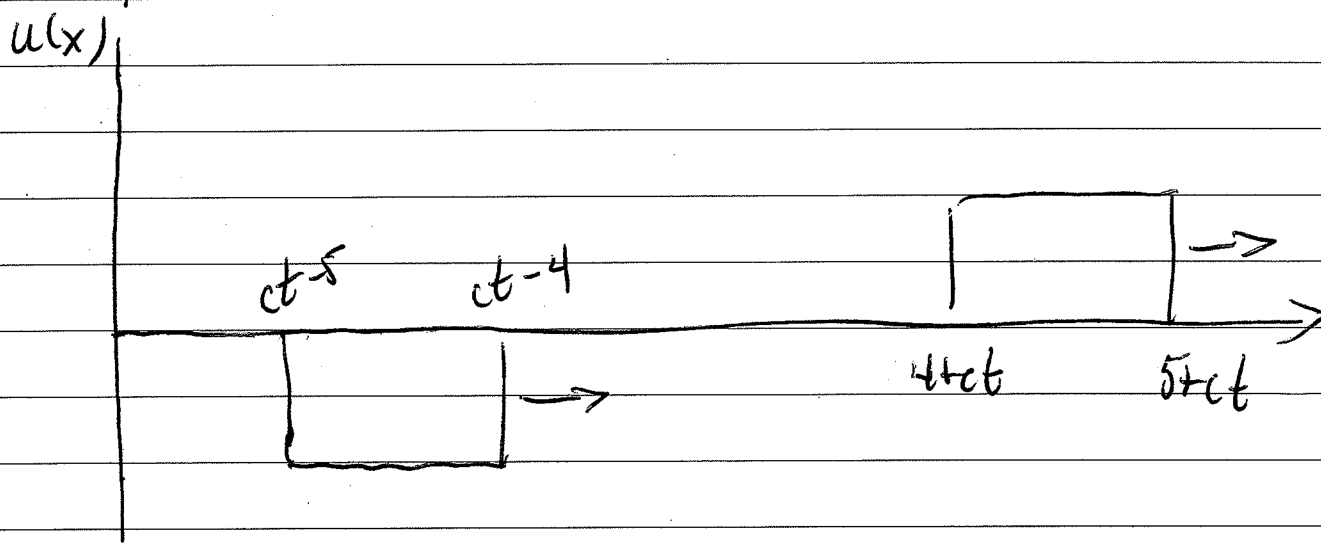




ct-4
5-ct
4+ct
5+ct



t = 5/c
ct-5
ct-4
4+ct
5+ct



u(x)
ct-5
ct-4
4+ct
5+ct

Method of Images