Math 322 Lecture 4

Topics

1. Boundary conditions in 2D, 3D
2. Review terminology: "linear," "homogeneous"

   Significance of linearity; why is it important?
3. The Method of Separation for

   linear, homogeneous PDEs with linear, homogeneous boundary conditions

   start today, finish Friday

Boundary Conditions in Higher Dimensions

1. Dirichlet $\quad T(\underline{x},t)\Big|_{\Omega} = T_{\Omega}(t) \quad$ (given)

$T(\underline{x},t)\Big|_{\Omega}$ means $T(\underline{x},t)$ evaluated on the boundary Ω

2. Neumann / Flux

$$-k_0 \underline{\nabla} T(\underline{x},t)\Big|_{\Omega} \cdot (-\hat{n}) = \phi_{\Omega}(t) \quad \text{(given)}$$

Now $\underline{\nabla} T(\underline{x},t)$ is evaluated on Ω

$$\Rightarrow \quad \underline{\nabla} T(\underline{x},t)\Big|_{\Omega} \cdot \hat{n} = \phi^{*}_{\Omega}(t) \quad \text{(given)}$$

3. Newton's Law of Cooling / Robin

"Flux through the boundary is proportional to the difference between the temp at the boundary and the temp in the bath"

$$-k_0 \underline{\nabla} T(\underline{x},t)\Big|_{\Omega} \cdot (-\hat{n}) = H\left(T_B(t) - T(\underline{x},t)\Big|_{\Omega}\right)$$

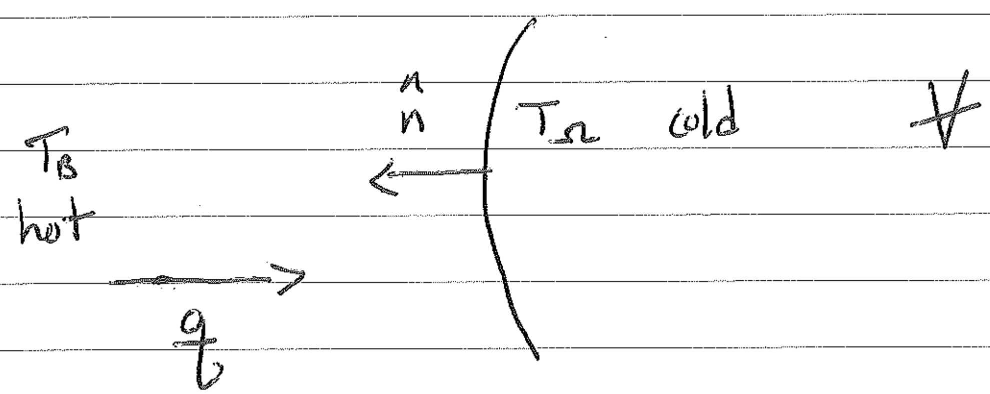
$T_B(t)$ temp. of bath is known/given, $H > 0$

Shorthand: $-k_0 \nabla T\big|_{\Omega} \cdot (-\hat{n}) = H \left(T_B - T\big|_{\Omega}\right)$, or

$$k_0 \underline{\nabla} T\big|_{\Omega} \cdot \hat{n} + H T\big|_{\Omega} = H T_B \qquad \text{(given)}$$

We know heat/energy is entering the domain when LHS > 0. This happens when

$T_B > T_\Omega$ ✓



$\underline{q} = -k_0 \underline{\nabla} T$ $\qquad \underline{q} \cdot (-\hat{n}) > 0 \Rightarrow$ heat entering the $\mathcal{V}$

$$-k_0 \underline{\nabla} T\big|_{\Omega} \cdot (-\hat{n}) > 0$$

$$H \left(T_B - T\big|_{\Omega}\right) > 0 \quad \text{for } H > 0$$

$\boxed{\text{Check for } T_\Omega > T_B \text{ !}}$ Do at home

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The Heat Eqn. is linear $\Rightarrow$ the dependent variable and all of its derivatives appear linearly

$$\frac{\partial T}{\partial t} = K_0 \frac{\partial^2 T}{\partial x^2} + Q \quad \text{is linear}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f \quad \text{is nonlinear}$$

Now lets switch over to $u(x,t)$ for the dependent variable, even for the heat eqn:

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = f \quad \text{or} \quad L(u) = f$$

$$L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} \quad \text{is linear}$$

compared to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = f \quad \text{or} \quad NL(u) = f$$

$$NL = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} - \nu\frac{\partial^2}{\partial x^2} \quad \text{is nonlinear}$$

A linear operator L satisfies

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

c_1, c_2 constants

Check $\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)\left(c_1 u_1(x,t) + c_2 u_2(x,t)\right)$

$$= \underset{\text{algebra}}{\ldots} = c_1\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)u_1(x,t)$$

$$+ c_2\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)u_2(x,t)$$

It is NOT true in general for a nonlinear operator

$$NL(c_1 u_1 + c_2 u_2) \neq c_1 NL(u_1) + c_2 NL(u_2)$$

Consider the nonlinear operator of squaring

$$(u_1 + u_2)^2 \neq u_1^2 + u_2^2 \quad !$$

Principle of Superposition If u_1 and u_2 satisfy a linear homogeneous equation, then the linear combination $c_1 u_1 + c_2 u_2$ satisfies the same linear homogeneous equation

$\Rightarrow$ Given 2 solutions u_1, u_2, we can find new solutions $c_1 u_1 + c u_2$

Recall homogeneous means $F = 0$

The concepts of linearity and homogeneity also apply to boundary conditions

$\frac{\partial u}{\partial x}(0,t) = g(t)$ is linear, non-homogeneous

$-K_0 \frac{\partial u}{\partial x}(L,t) - H\, u(L,t) = 0$ is

linear homogeneous

$-K_0 \frac{\partial u}{\partial x}(L,t) - H\, u(L,t) = h(t)$ is

linear, non-homogeneous

Separation of Variables

Linear homogeneous equations ; linear homogeneous b.c.s

Example

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L \, , \quad t > 0$$

1D Heat Eqn ; homogeneous

$u(0,t) = 0$
$u(L,t) = 0$ homogeneous (zero) Dirichlet b.c.s

$u(x,0) = F(x)$ initial condition

First lets work with the equation ; Try

$$u(x,t) = \phi(x)\, G(t) \qquad (\text{separate } x \, , \; t)$$

Plug in : $\displaystyle \phi(x) \frac{dG(t)}{dt} = k\, G(t) \frac{d^2\phi(x)}{dx^2}$

Formally multiply both sides by $\displaystyle \frac{1}{k\,\phi(x)\,G(t)}$

(not worrying about divide by zero !)

$$\frac{1}{k}\frac{1}{G(t)}\frac{dG(t)}{dt} = \frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2}$$

LHS is a function of t ; RHS a function of x

$\Rightarrow$ LHS = RHS = Constant = $-\lambda$

where the minus sign is convention / convenience

$$\Rightarrow \quad \frac{dG(t)}{dt} = -\lambda k G(t)$$

$$\frac{d^2\phi(x)}{dx^2} = -\lambda \phi(x)$$

Our ansatz $u(x,t) = \phi(x)\,G(t)$ reduces the PDE to ODEs (2) ; Now solve ODEs !

What happens to the boundary conditions ? Initial Cond ?

$u(0,t) = \phi(0)\,G(t) = 0$

$u(L,t) = \phi(L)\,G(t) = 0$

$u(x,0) = \phi(x)\,G(0) = f(x)$