Lecture 5 Math 322

Separation of Variables for

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0$$

1D homogeneous heat eqn.

$u(0,t) = u(L,t) = 0$ homogeneous Dirichlet b.c.s

$u(x,0) = f(x)$ initial condition

Ansatz $u(x,t) = \phi(x)\,G(t)$ followed by algebra (formal) $\Rightarrow$

$$\frac{1}{k}\frac{1}{G(t)}\frac{dG(t)}{dt} = \frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2} \qquad \left(= -\lambda \right)$$

$\Rightarrow$ 2 ODEs:

$$\frac{dG(t)}{dt} = -\lambda k\, G(t)$$

$$\frac{d^2\phi(x)}{dx^2} = -\lambda\, \phi(x)$$

What happens to the boundary and initial conditions?

$u(0,t) = \phi(0)\,G(t) = 0 \qquad u(L,t) = \phi(L)\,G(t) = 0$

$u(x,0) = \phi(x)\,G(0) = f(x)$

Choosing $G(t) = 0 \Rightarrow u(x,t) = 0$ the "trivial solution"

Looking for non-zero, non-trivial solutions leads to

$$\phi(0) = \phi(L) = 0 \quad ; \quad \phi(x) G(0) = F(x)$$

Now we see that the x-dependence is completely determined by the boundary value problem

$$\frac{d^2\phi(x)}{dx^2} = -\lambda \phi(x) \qquad \phi(0) = 0 \qquad \phi(L) = 0$$

2^{nd}-order equ. (ODE) 2 conditions

and we need to find all possible values of λ (the eigenvalues) and $\phi_\lambda(x)$ (the eigenfunctions) corresponding to λ {For now assume λ real; later prove λ real}.

Lets try all possibilities

$\boxed{\lambda = 0} \Rightarrow \frac{d^2\phi(x)}{dx^2} = 0 \qquad \phi(0) = \phi(L) = 0$

$\Rightarrow \phi(x) = Ax + B, \quad \phi(0) = \phi(L) = 0$

$\phi(0) = 0 \Rightarrow B = 0 \quad ; \quad \phi(L) = 0 \Rightarrow A = 0$

trivial solution

$\boxed{\lambda < 0}$ $\phi(x) = Ae^{rx}$ {why?} $\Rightarrow$

$r^2 Ae^{rx} = -\lambda Ae^{rx} \Rightarrow r^2 = -\lambda > 0$

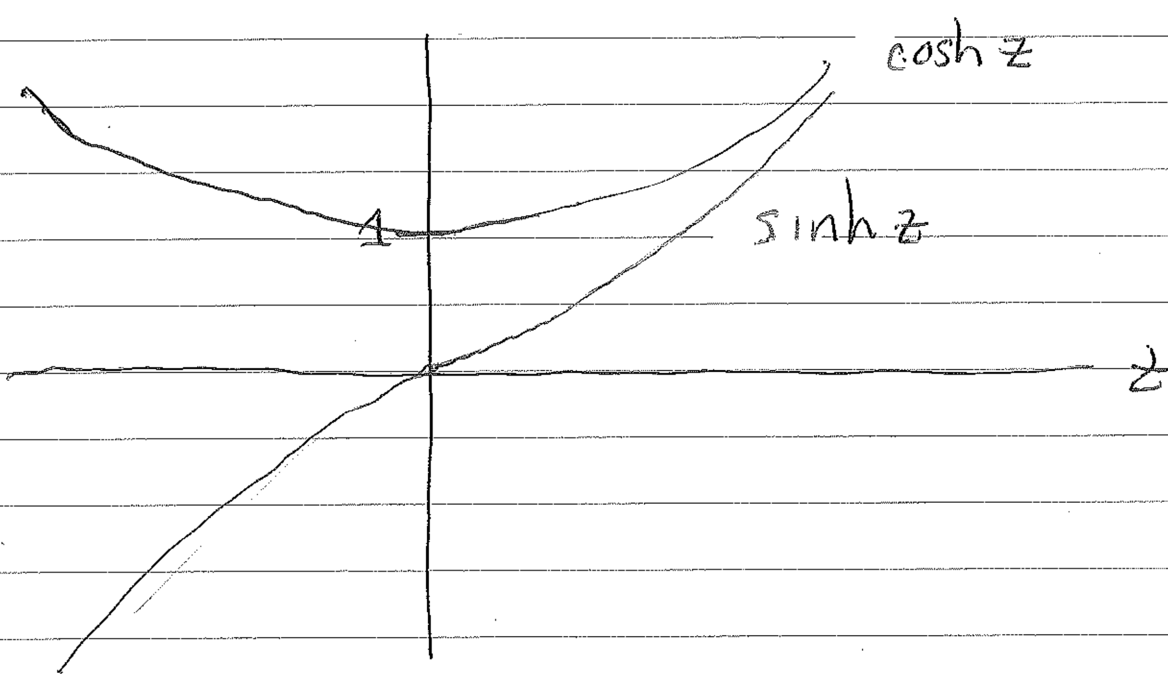
$r_{1,2} = \pm\sqrt{-\lambda} = \pm s \qquad s > 0$

$\phi(x) = A_1 e^{sx} + A_2 e^{-sx}$ $\boxed{\text{or equivalently, by Principle of Superposition}}$

$\phi(x) = C_1 \cosh sx + C_2 \sinh sx$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

$z = sx$ here



Now apply boundary conditions

$$\phi(0) = 0 \implies C_1 = 0$$

$$\phi(L) = C_2 \sin(\sqrt{-\lambda}\, L) \neq 0 \quad \text{for } L > 0$$

so the b.c. at $x = L$ cannot be satisfied

$$\implies \lambda \nless 0$$

$\boxed{\lambda > 0}$ $\quad \phi(x) = Ae^{rx} \implies r^2 = -\lambda < 0$

$$\implies \phi(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

$$\phi(0) = 0 \implies C_1 = 0$$

$$\phi(L) = 0 \implies C_2 \sin(\sqrt{\lambda}\, L) = 0$$

$$\implies \sqrt{\lambda}\, L = n\pi$$

$n = 1, 2, 3 \ldots$ integer
$n = 0$ excluded
$n < 0$ excluded

$$\implies \lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, 3, \ldots$$

$$\phi_\lambda(x) = C_2 \sin(\sqrt{\lambda_n}\, x) \qquad \text{or}$$

$$\phi_n(x) = C_2 \sin\left(\frac{n\pi}{L} x\right)$$

Remarks

* λ_n are called the eigenvalues ; $\phi_n(x)$ are the eigenfunctions

* There are an infinite number of eigenvalues, starting with the smallest and with value tending to ∞ as $n \to \infty$

* For each eigenvalue, there is one eigenfunction

* By Principle of Superposition

$$\phi(x) = \sum_{\lambda} b_\lambda \phi_\lambda(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

In this case $\phi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

Satisfying $\frac{d^2\phi(x)}{dx^2} = -\lambda \phi(x) \quad 0 < x < L$

$$\phi(0) = 0 \qquad \phi(L) = 0$$

But this is only the x-dependence of the problem ...