Math 322 Lecture 7

1D heat eqn. with periodic boundary conditions

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad -L < x < L$$

$$u(-L,t) = u(L,t) \qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=-L} = \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=L}$$

continuity of temp.

derivative of temp. wrt x is also continuous

Method of Separation $u(x,t) = \phi(x)G(t)$ gives the same 2 ODEs.

$$\frac{d^2\phi(x)}{dx^2} = -\lambda\,\phi(x) \qquad \frac{dG(t)}{dt} = -\lambda k G(t)$$

Boundary Conditions:

$$u(-L,t) = u(L,t) \Rightarrow \phi(-L)G(t) = \phi(L)G(t)$$

$$\Rightarrow \phi(-L) = \phi(L)$$

$$\frac{\partial u(-L,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} \Rightarrow \left.\frac{d\phi(x)}{dx}\right|_{x=-L} = \left.\frac{d\phi(x)}{dx}\right|_{x=L}$$

(shorthand)

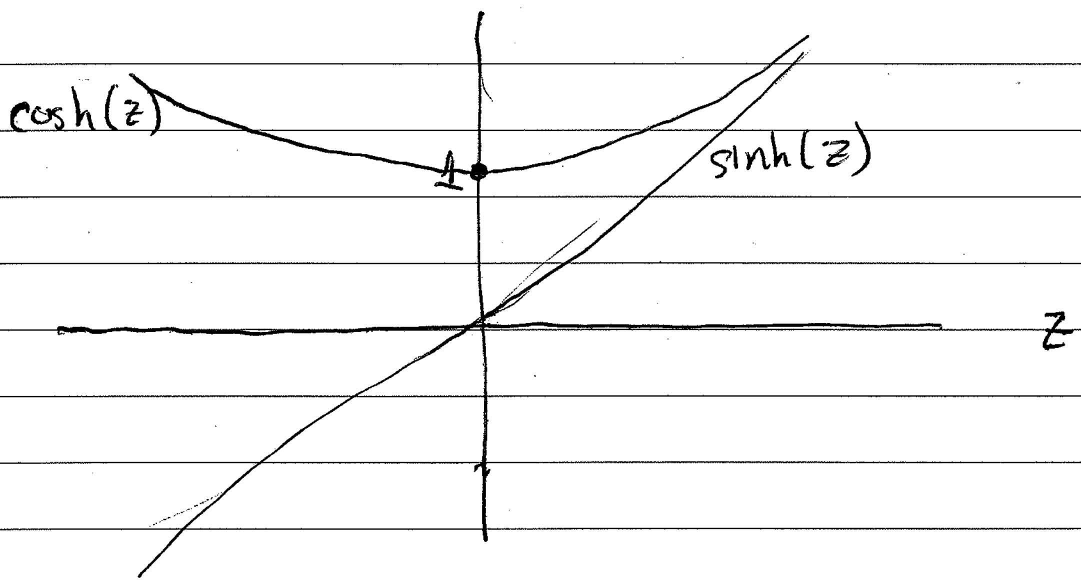
$$\phi'(-L) = \phi'(L)$$

Lets leave the initial condition for the end

Boundary value problem for x-dependence

$$\frac{d^2\phi(x)}{dx^2} = -\lambda\phi(x) \qquad \phi(-L) = \phi(L) \qquad \phi'(-L) = \phi'(L)$$

$\boxed{\lambda < 0}$ $\phi(x) = C_1 \cosh(sx) + C_2 \sinh(sx)$ $\quad s = \sqrt{-\lambda} > 0$



$\phi(-L) = \phi(L) \Rightarrow$ $C_1 \cosh(-Ls) + C_2 \sinh(-Ls)$

$= C_1 \cosh(Ls) + C_2 \sinh(Ls)$

$\cosh(-Ls) = \underset{\text{(even)}}{\cosh(Ls)} \Rightarrow$

$C_2 \sinh(-Ls) = C_2 \sin(Ls)$ only true if $C_2 = 0$

Since $L \neq 0$, $s \neq 0$

Then $\phi(x) = C_1 \cosh(sx)$

Now try to impose $\phi'(-L) = \phi'(L)$

$$\frac{d\phi}{dx} = s\,C_1 \sinh(sx)$$

$$\rightarrow \quad s\,C_1 \sinh(-sL) = s\,C_1 \sinh(sL)$$

but $\sinh(-sL) \neq \sinh(sL)$ unless $L=0$ or $s=0$
and we have $L>0$ $s=\sqrt{-\lambda} > 0$

$$\Longrightarrow \quad C_1 = 0$$

Thus $\lambda < 0$ cannot satisfy periodic boundary conditions with a non-trivial solution

$\boxed{\lambda = 0}$ $\phi(x) = C_1 x + C_2$

$\phi(-L) = \phi(L) \Rightarrow C_1(-L) + C_2 = C_1(L) + C_2$

$\Rightarrow C_1 = 0$

Then $\phi(x) = C_2$ ok for $\left.\frac{d\phi}{dx}\right|_{x=-L} = \left.\frac{d\phi}{dx}\right|_{x=L} = 0$!

So $\phi(x) = C_2$ for $\lambda = 0$ works.

$\boxed{\lambda > 0}$ $\phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$\phi(-L) = \phi(L)$: $C_1 \cos(-L\sqrt{\lambda}) + C_2 \sin(-L\sqrt{\lambda})$

$= C_1 \cos(L\sqrt{\lambda}) + C_2 \sin(L\sqrt{\lambda})$

$\cos(-L\sqrt{\lambda}) = \cos(L\sqrt{\lambda})$ since cosine is even

$\Rightarrow C_2 \sin(-L\sqrt{\lambda}) = C_2 \sin(L\sqrt{\lambda})$ can be satisfied

by $\sqrt{\lambda} = \frac{n\pi}{L}$ $n = 1, 2, 3 \ldots$

instead of $C_2 = 0$

Notice that we are using $\sqrt{\lambda} > 0$ since we've taken out $\pm i$

Lets leave $C_2 \neq 0$, $\sqrt{\lambda} = \frac{n\pi}{L}$ $n = 1, 2, 3 \ldots$

and see what happens

$$\phi(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

$$\frac{d\phi(x)}{dx} = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda}\, x) + \sqrt{\lambda}\, C_2 \cos\sqrt{\lambda}\, x$$

$$\left.\frac{d\phi(x)}{dx}\right|_{x=-L} = \left.\frac{d\phi(x)}{dx}\right|_{x=L} \Rightarrow$$

$$-\sqrt{\lambda}\, C_1 \sin(-L\sqrt{\lambda}) + \sqrt{\lambda}\, C_2 \cos(-L\sqrt{\lambda}) =$$

$$-\sqrt{\lambda}\, C_1 \sin(L\sqrt{\lambda}) + \sqrt{\lambda}\, C_2 \cos(L\sqrt{\lambda})$$

since $\cos(-L\sqrt{\lambda}) = \cos(L\sqrt{\lambda})$ even

and $\sin(-L\sqrt{\lambda}) = \sin(L\sqrt{\lambda}) = 0$ $\sqrt{\lambda} = \frac{n\pi}{L}$

$n = 1, 2, 3 \ldots$

so the derivative condition is automatically satisfied with $C_1 \neq 0$, $C_2 \neq 0$ as long as $\sqrt{\lambda} = \frac{n\pi}{L}$ $n = 1, 2, 3 \ldots$

So we arrive at

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Since the time dependence is the same:

$$\frac{dG}{dt} = -k\lambda\, G(t) \qquad u(x,0) = \phi(x)\, G(0) = f(x)$$

$$G(t) = C \exp(-k\lambda t) \implies$$

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right) \exp\left(-k\frac{n^2\pi^2}{L^2} t\right)$$

or

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} \exp\left(-k\frac{n^2\pi^2}{L^2} t\right)$$

$$+ \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \exp\left(-k\frac{n^2\pi^2}{L^2} t\right)$$

Now for $\lambda_0 = 0$ $\quad \phi_0(x) = a_0$

$\lambda_n = \frac{n\pi}{L}$ $\quad n = 1, 2, 3 \ldots$ there are 2

linearly independent eigenfunctions $\cos\frac{n\pi x}{L}$, $\sin\frac{n\pi x}{L}$

The latter is something different for periodic boundary conditions compared to linear, homogeneous, separated boundary conditions (Dirichlet, Neumann, Robin)

$\boxed{\text{Orthogonality}}$

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$$

as can be checked by direct substitution

Note the domain is $[-L, L]$

Initial Condition $u(x,0) = f(x) =$ $\star\star$

$$\sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Multiply both sides by $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L; interchange the order of summation and integration

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\, f(x)\, dx$$

$$= \int_{-L}^{L} \left(\sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} \right) dx$$

$$= \sum_{n=0}^{\infty} \int_{-L}^{L} a_n \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx + \sum_{n=1}^{\infty} \int_{-L}^{L} b_n \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx$$

$$= \sum_{n=0}^{\infty} \int_{-L}^{L} a_n \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx + 0$$

only the integral with $n=m$ survives, and we have 2 cases

$n = m = 0$

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} a_0\,dx = a_0 x\Big|_{-L}^{L} = 2La_0$$

$$\Rightarrow \quad a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$n = m \neq 0$

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L} a_m \cos^2\left(\frac{m\pi x}{L}\right)dx = a_m L$$

$$\Rightarrow \quad a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad m \neq 0,\ m = 1, 2, 3 \ldots$$

Similarly, start from (**) on page 8; multiply both sides by $\sin\frac{m\pi x}{L}$ and integrate $\Rightarrow$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad m = 1, 2, 3 \ldots$$