Thus, the eigenfunctions are

$$y_n(x) \sim \left(\int_0^\pi \frac{\sqrt{Q(t)}}{2}\,dt\right)^{-1/2} Q^{-1/4}(x)\sin\left[n\pi\frac{\int_0^x \sqrt{Q(t)}\,dt}{\int_0^\pi \sqrt{Q(t)}\,dt}\right], \qquad n\to\infty. \tag{10.1.33}$$

Note that if $Q(x) \equiv 1$, then the right side of (10.1.33) reduces to $\sqrt{2/\pi}\sin(nx)$, which is the exact solution to $y'' + y = 0$ $[y(0) = y(\pi) = 0]$.

To demonstrate the accuracy of our results, we choose $Q(x) = (x + \pi)^4$. Then the approximate eigenvalues and eigenfunctions are given by

$$E_n \sim \frac{9n^2}{49\pi^4}, \qquad n\to\infty, \tag{10.1.34}$$

and

$$y_n(x) \sim \sqrt{\frac{6}{7\pi^3}}\,\frac{\sin\left[n(x^3 + 3x^2\pi + 3\pi^2 x)/7\pi^2\right]}{(\pi + x)}, \qquad n\to\infty. \tag{10.1.35}$$

We have checked these results numerically by computer. The comparisons between the approximate analytical and the computer solutions are given in Table 10.1 and Figs. 10.2 and 10.3.

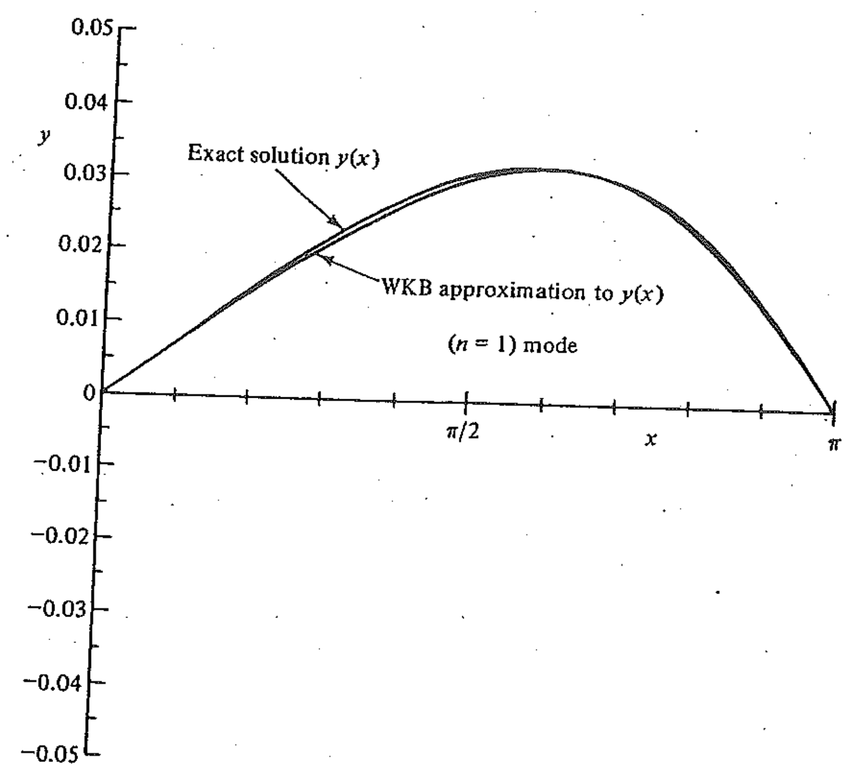

Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x + \pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$, with the WKB approximation to this solution as given in (10.1.35) for the lowest $(n = 1)$ mode. Although WKB becomes exact as $n \to \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x+\pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$ with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ $(n \to \infty)$

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate − exact)/(exact)

n	E_n(WKB)	E_n(exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

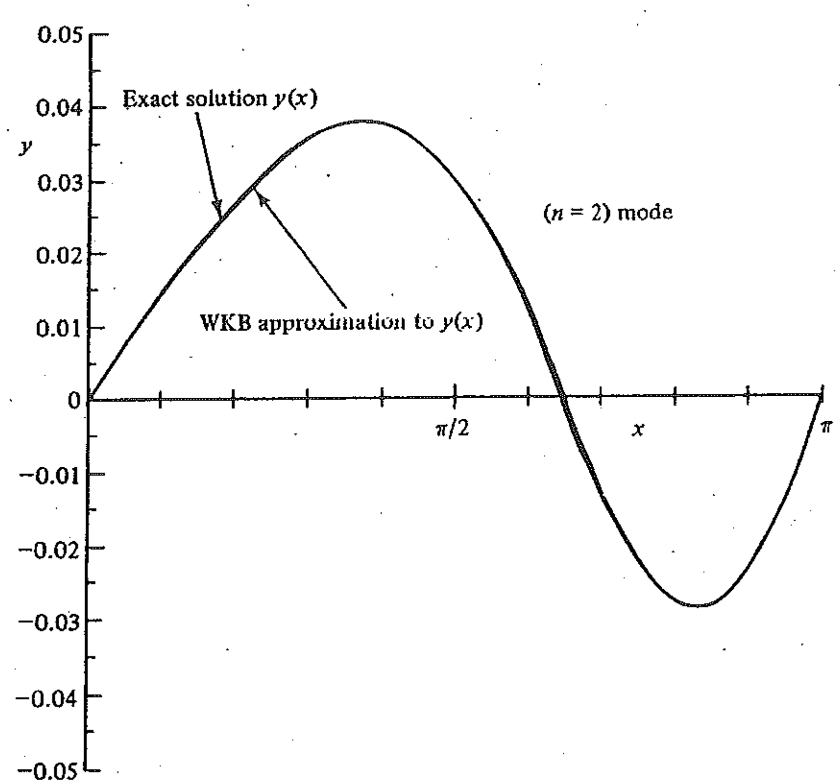

Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.