1 Tuesday, November 4

Last time we showed that if

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{} 0$$

is a short exact sequence of chain complexes, then there is an associated long exact sequence for homology

$$\cdots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \rightarrow \cdots$$

Corollary 1. Long exact sequence for homology of a pair \((X, A)\):

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots$$

Corollary 2. Long exact sequence for reduced homology of a pair \((X, A)\):

$$\cdots \rightarrow \tilde{H}_n(A) \rightarrow \tilde{H}_n(X) \rightarrow \tilde{H}_n(X, A) \rightarrow \tilde{H}_{n-1}(A) \rightarrow \cdots$$

The long exact sequence for reduced homology of a pair \((X, A)\) is the long exact sequence associated to the augmented short exact sequence for \((X, A)\):

$$\begin{array}{cccccc}
0 & \rightarrow & C_s(A) & \rightarrow & C_s(X) & \rightarrow & C_s(X, A) & \rightarrow & 0 \\
& & \downarrow\varepsilon & & \downarrow\varepsilon & & \downarrow0 & & \\
0 & \rightarrow & Z & \rightarrow & Z & \rightarrow & 0 & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}$$

If \(x_0 \in X\), the long exact sequence for reduced homology of the pair \((X, x_0)\) gives us \(\tilde{H}_n(X) \cong H_n(X, x_0)\) for all \(n\).
Corollary 3. Long exact sequence for homology of a triple \((X, A, B)\) where \(B \subseteq A \subseteq X\):

\[
\cdots \rightarrow H_n(A, B) \rightarrow H_n(X, B) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A, B) \rightarrow \cdots
\]

**Proof.** Start with the short exact sequence of chain complexes

\[
0 \rightarrow C_\ast(A, B) \rightarrow C_\ast(X, B) \rightarrow C_\ast(X, A) \rightarrow 0
\]

where maps are induced by inclusions of pairs. Take the associated long exact sequence for homology. \(\square\)

We next discuss chain maps induced by maps of pairs of spaces. Let \(f : (X, A) \rightarrow (Y, B)\) be a map \(f : X \rightarrow Y\) such that \(f(A) \subseteq B\). Then \(f\) induces \(f_n : C_n(X) \rightarrow C_n(Y)\) so that \(f_n(C_n(A)) \subseteq C_n(B)\) for all \(n\). So we get an induced homomorphism \(f_\ast : C_\ast(X, A) \rightarrow C_\ast(Y, B)\). Because \(f_n\) satisfies the boundary operator \(\partial\), it also holds for the induced maps on the quotients. Therefore we get induced homomorphisms on homology \(f_\ast : H_\ast(X, A) \rightarrow H_\ast(Y, B)\) for all \(n\).

**Proposition 1.** If \(f, g : (X, A) \rightarrow (Y, B)\) are homotopic through maps of pairs \((X, A) \rightarrow (Y, B)\), then \(f_\ast = g_\ast : H_\ast(X, A) \rightarrow H_\ast(Y, B)\) for all \(n\).

**Proof.** The prism operator \(P : C_n(X) \rightarrow C_{n+1}(Y)\), which satisfies \(\partial P + P\partial = g_\ast - f_\ast\), takes \(C_n(A)\) into \(C_{n+1}(B)\) by construction. So we get a prism operator on quotients \(P : C_n(X, A) \rightarrow C_{n+1}(Y, B)\) which satisfies \(\partial P + P\partial = g_\ast - f_\ast\) on \(C_n(X, A)\). Hence \(f_\ast\) and \(g_\ast\) have the same effect on \(H_\ast(X, A)\) for all \(n\). That is, \(f_\ast = g_\ast : H_\ast(X, A) \rightarrow H_\ast(Y, B)\) for all \(n\). \(\square\)

**Theorem 1.** (Excision Theorem) Given subspaces \(Z \subseteq A \subseteq X\) so that \(\overline{Z} \subseteq \text{int}(A)\), the inclusion \((X \setminus Z, A \setminus Z) \rightarrow (X, A)\) induces isomorphisms \(H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A)\) for all \(n\). Equivalently, if \(A, B \subseteq X\) are such that \(X = \text{int}(A) \cup \text{int}(B)\), the inclusion \((B, A \cap B) \hookrightarrow (X, A)\) induces isomorphisms \(H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)\) for all \(n\).

To see that the two statements of the Excision Theorem are equivalent, just take \(B = X \setminus Z\) (or \(Z = X \setminus B\)). Then \(A \cap B = A \setminus Z\) and \(\overline{Z} \subseteq \text{int}(A) \iff X = \text{int}(A) \cup \text{int}(B)\).

Before proving the Excision Theorem, we will observe some of its applications. For example we have the Suspension Theorem for homology. For a space \(X\), define \(\Sigma X\) to be the quotient of \(X \times [-1, 1]\) obtained by identifying \(X \times \{-1\}\) to one point and \(X \times \{1\}\) to another point. For example, if \(X = S^n\), then \(\Sigma X \cong S^{n+1}\).

**Theorem 2.** (Suspension Theorem) \(\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Sigma X)\) for all \(i \geq 0\).

**Corollary 4.** \(\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = n \\ 0, & \text{if } i \neq n \end{cases}\)
Proof of Corollary. (Assuming Suspension Theorem) We use induction. \( \tilde{H}_0(S^0) \cong \mathbb{Z} \) because \( S^0 \) is two points. For \( i > 0 \), \( \tilde{H}_i(S^0) = H_i(S^0) = H_i(\{−1\}) \oplus H_i(\{1\}) \cong 0 \). So the statement holds for \( S^0 \). Assume it holds for \( S^n \). If \( i = 0 \), we know \( \tilde{H}_i(S^{n+1}) \cong 0 \) because \( S^{n+1} \) is connected. If \( i > 0 \), then \( \tilde{H}_i(S^{n+1}) = \tilde{H}_{i−1}(S^n) \) by the Suspension Theorem. If \( i = n + 1 \) then this is \( \cong \mathbb{Z} \). If \( i \neq n + 1 \) then this is \( \cong 0 \). \( \square \)

Proof of Suspension Theorem. (Assuming Excision Theorem). Let \( \pi : X \times [−1, 1] \to \Sigma X \) be the quotient map identifying \( \Sigma X \). Given a topological space \( X \), let \( \tilde{H}_i(\Sigma X, S) \cong H_i(\Sigma X, S) \). Then we have the following:

1. \( \tilde{H}_i(\Sigma X) \cong H_i(\Sigma X, S) \).
2. \( H_i(\Sigma X, S) \cong H_i(\Sigma X, \Sigma X) \). This can be seen in two ways:
   a. We observe that \( \Sigma X \) deformation retracts to \( S \) and apply homotopy invariance for the homology of a pair.
   b. \( H_i(\Sigma X, S) \cong 0 \) by the long exact sequence for reduced homology of the pair \( (\Sigma X, S) \). Then the long exact sequence for homology of the triple \( (\Sigma X, \Sigma X, S) \) gives us 2.
3. \( H_i(\Sigma X, \Sigma X) \cong H_i(\Sigma X, X) \). This can be seen if we excise \( \text{int}(\Sigma X) \) then apply the Excision Theorem and homotopy invariance for the homology of a pair.
4. \( H_i(\Sigma X, X) \cong \tilde{H}_{i−1}(X) \) by applying the long exact sequence for reduced homology of the pair \( (\Sigma X, X) \) and the fact that \( \Sigma X \) is contractible. \( \square \)

(Sketch for) Proof of Excision Theorem. Given a topological space \( X \), let \( \mathcal{U} = \{U_j\}_j \) be a collection of subspaces of \( X \) whose interiors cover \( X \). Let \( C^\mathcal{U}_n(X) = \{\sum_{i=1}^m n_i \sigma_i : m \in \mathbb{N}_{\geq 1}, n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \) is such that there exists \( j \) with \( \sigma_i(\Delta^n) \subseteq U_j \}. \) Then \( C^\mathcal{U}_n(X) \leq C_n(X) \). Furthermore, \( \partial_n : C_n(X) \to C_{n−1}(X) \) induces boundary maps \( \partial_n \) on \( C^\mathcal{U}_n(X) \) satisfying \( \partial^2 = 0 \). So we get a chain complex \( (C^\mathcal{U}_n(X), \partial_n) \) whose \( n \)-th homology group is denoted by \( H^\mathcal{U}_n(X) \).

By subdividing simplices in \( X \), it can be shown that the map \( H^\mathcal{U}_n(X) \to H_n(X) \) induced by the inclusion is an isomorphism for all \( n \). In fact, the inclusion \( i : C^\mathcal{U}_n(X) \to C_n(X) \) is a chain homotopy equivalence. That is, there exists a chain map \( \rho : C_n(X) \to C^\mathcal{U}_n(X) \) such that \( i \rho \) and \( \rho i \) (the latter of which is precisely the identity map) are both chain homotopic to the identity map. So there exists \( P : C_n(X) \to C_{n+1}(X) \) such that \( \partial P + P \partial = \text{id} − i \rho \).

In the case of the Excision Theorem, \( \mathcal{U} = \{A, B\} \). We let \( C_n(A+B) \) denote \( C^\mathcal{U}_n(X) \). Every operator appearing in \( \partial P + P \partial = \text{id} − i \rho \) takes chains in \( A \) to chains in \( A \), so we can factor out the chains in \( A \) to conclude that the inclusions \( C_n(A+B)/C_n(A) \to C_n/X/C_n(A) = C_n(X, A) \) also induce isomorphisms on homology. But the map \( C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \to C_n(A + B)/C_n(A) \) induced by the inclusion is an isomorphism by the first isomorphism theorem. Combining these statements, we obtain isomorphisms \( H_n(B, A \cap B) \cong H_n(X, A) \). \( \square \)
2 Thursday, November 6

**Theorem 3.** (Brower) If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are nonempty homeomorphic open sets, then $m = n$.

**Proof.** For all $x \in U$ and for all $k$, we have $H_k(U, U - \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ by applying the second version of the Excision Theorem with $X = \mathbb{R}^m$, $B = U$, and $A = \mathbb{R}^m - \{x\}$. Combining this with the long exact sequence for reduced homology of the pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ and the fact that $\mathbb{R}^m - \{x\}$ is homotopy equivalent to $S^{m-1}$, we obtain for all $x \in U$, for all $k$:

$$H_k(U, U - \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong H_{k-1}(\mathbb{R}^m - \{x\}) \cong H_{k-1}(S^{m-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}$$

Likewise, if $y \in V$, we have for all $k$:

$$H_k(V, V - \{y\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

But if $f : U \to V$ is a homeomorphism, then $f : U - \{x\} \to V - \{f(x)\}$ is a homeomorphism. Hence $f$ induces isomorphisms $H_k(U, U - \{x\}) \cong H_k(V, V - \{f(x)\})$ for all $k$. Therefore, $m = n$. \hfill \square

**Remark.** If $X$ is a topological space, $x \in X$, and $U \subseteq X$ is an open neighborhood of $x$, then for all $n$, by the Excision Theorem, $H_n(X, X - \{x\}) \cong H_n(U, U - \{x\})$. In particular, for all $n$, $H_n(X, X - \{x\})$ depends only on the topology of a neighborhood of $x$. Therefore these homology groups are called the local homology groups (at $x$). They can be used to check when a map $f : X \to Y$ is not a local homeomorphism.

**Theorem 4.** (Brower’s Fixed Point Theorem)

(i) $\partial D^n$ is not a retract of $D^n$.

(ii) Any continuous map $f : D^n \to D^n$ has a fixed point.

**Proof.** (i): Assume towards a contradiction that there exists a retraction $r : D^n \to \partial D^n = S^{n-1}$, then, if $i : S^{n-1} \to D^n$ is the inclusion, we have $r \circ i = \text{id}_{S^{n-1}}$. By functoriality, for all $k$ we have $(r \circ i)_* = r_* \circ i_* = \text{id}_{H_k(S^{n-1})}$. If $k = n - 1$ we obtain:

$$\mathbb{Z} \xrightarrow{\cong} H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(S^{n-1}) \xrightarrow{\cong} \mathbb{Z}$$

$$\circ \quad \text{id}_\mathbb{Z}$$

But $r_* = 0$ and $i_* = 0$ because $H_{n-1}(D^n) = 0$. Therefore we have arrived at a contradiction.
(ii): Let \( f : D^n \to D^n \) be a continuous map. Assume towards a contradiction that \( f(x) \neq x \) for all \( x \in D^n \). Then we may define a function \( r : D^n \to S^{n-1} \) in the following way. Let \( x \in D^n \) and let \( [f(x), x] \) denote the (unique) ray based at \( f(x) \) passing through \( x \). Define \( r(x) \) to be the unique element in \( ([f(x), x]) \cap \partial D^n - \{f(x)\} \). Then \( r \) is continuous and is a retraction \( D^n \to \partial D^n \), contradicting (i). \( \square \)

**Theorem 5.** (Mayer with an "a" - Vietoris Sequence) Suppose \( X = A \cup B = \text{int}(A) \cup \text{int}(B) \). Then there is a long exact sequence:

\[
\cdots \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \oplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0
\]

**Proof.** Let \( C_n(A+B) \) denote the subgroup of \( C_n(X) \) whose elements are precisely sums of singular simplices in either \( A \) or \( B \). The boundary maps \( \partial \) on \( C_n(X) \) restrict to \( C_n(A+B) \) and we get a chain complex \( (C_*(A+B), \partial) \) whose homology is isomorphic to the homology of \( X \) (this was verified last time). Hence, we need only produce a long exact sequence of the form specified in the theorem where each \( H_n(X) \) is replaced by the \( n \)th homology group of \((C_*(A+B), \partial_*))\). To this end, for \( n \in \mathbb{N}_{\geq 0} \), consider the following sequence:

\[
0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0 \tag{\ast}
\]

where, \( \phi(x) = (x, -x) \) for all \( x \in C_n(A \cap B) \) and \( \psi(x,y) = x + y \) for all \( (x,y) \in C_n(A) \oplus C_n(B) \). I claim that this sequence is exact:

- \( \psi \) is surjective by the definition of \( C_n(A+B) \).
- \( \phi \) is injective.
- For all \( x \in C_n(A \cap B) \), \( \psi \circ \phi(x) = x - x = 0 \). Therefore \( \text{im}(\phi) \subseteq \ker(\psi) \).
- If \( (x,y) \in \ker(\psi) \), then \( x \) is a chain in \( A \), \( y \) is a chain in \( B \), and \( y = -x \). This implies that \( x \) is a chain in \( A \cap B \) and \( \phi(x) = (x,-x) = (x,y) \). Therefore \( \ker(\psi) \subseteq \text{im}(\phi) \).

The Mayer-Vietoris Sequence is the long exact sequence associated to \( \ast \). \( \square \)

**Remark.** By using augmented chain complexes in \( \ast \), we also obtain a reduced Mayer-Vietoris Sequence.

**Example 1.** Let \( X = S^n \), \( A = S^n - \{S\} \), and \( B = S^n - \{N\} \) where \( S \) and \( N \) are the south pole and north pole respectively. Then \( A \cong \mathbb{R}^n \), \( B \cong \mathbb{R}^n \), and \( A \cap B \cong S^{n-1} \). From the reduced Mayer-Vietoris Sequence, we get \( \widetilde{H}_i(S^n) \cong \widetilde{H}_{i-1}(S^{n-1}) \) for all \( i \). By induction, we find as before:

\[
\widetilde{H}_i(S^n) \cong \begin{cases} 
\mathbb{Z}, & \text{if } i = n \\
0, & \text{if } i \neq n
\end{cases}
\]

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Example 2. (Homology of a Klein Bottle) Let $K$ be the Klein bottle. It may be decomposed into $K = M_1 \cup M_2$ where $M_1$ and $M_2$ are Mobius bands that are glued along their boundary circles. Depicted below.

Each of $M_1, M_2$ is homotopy equivalent to its boundary circle $S^1$. $M_1 \cap M_2 = S^1$ is that boundary circle. By the reduced Mayer-Vietoris Sequence, $H_n(K) \cong 0$ for all $n > 2$. Consider the segment of the reduced Mayer-Vietoris Sequence below:

$$
0 \longrightarrow H_2(K) \longrightarrow H_1(M_1 \cap M_2) \overset{\phi}{\longrightarrow} H_1(M_1) \oplus H_1(M_2) \overset{\psi}{\longrightarrow} H_1(K) \longrightarrow 0
$$

Then $\phi : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ maps 1 to $(2, -2)$. By exactness, $H_2(K) \cong \ker(\phi) \cong 0$ and $H_1(K) \cong \operatorname{Coker}(\phi) \cong (\mathbb{Z} \oplus \mathbb{Z})/(2(1, -1))$. If we consider the basis $\{(1, 0), (1, -1)\}$ of $\mathbb{Z} \oplus \mathbb{Z}$, then $(\mathbb{Z} \oplus \mathbb{Z})/(2(1, -1)) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. In sum we’ve concluded the following:

$$
\tilde{H}_i(K) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } i = 1 \\
0, & \text{if } i \neq 1
\end{cases}
$$