0.1 Euler Characteristic

Definition 0.1.1. Let $X$ be a finite CW complex of dimension $n$ and denote by $c_i$ the number of $i$-cells of $X$. The Euler characteristic of $X$ is defined as:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot c_i.$$ (0.1.1)

It is natural to question whether or not the Euler characteristic depends on the cell structure chosen for the space $X$. As we will see below, this is not the case. For this, it suffices to show that the Euler characteristic depends only on the cellular homology of the space $X$. Indeed, cellular homology is isomorphic to singular homology, and the latter is independent of the cell structure on $X$.

Recall that if $G$ is a finitely generated abelian group, then $G$ decomposes into a free part and a torsion part, i.e.,

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}.$$ 

The integer $r := \text{rk}(G)$ is the rank of $G$. The rank is additive in short exact sequences of finitely generated abelian groups.

Theorem 0.1.2. The Euler characteristic can be computed as:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot b_i(X)$$ (0.1.2)

with $b_i(X) := \text{rk}(H_i(X))$ the $i$-th Betti number of $X$. In particular, $\chi(X)$ is independent of the chosen cell structure on $X$.

Proof. We use the following notation: $B_i = \text{Image}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i = Z_i/B_i$.

Consider a (finite) chain complex of finitely generated abelian groups and the short exact sequences defining homology:

$$0 \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

$$0 \xrightarrow{\iota} Z_i \xrightarrow{d_i} C_i \xrightarrow{d_i} B_{i-1} \xrightarrow{q} H_i \xrightarrow{q} 0$$

The additivity of rank yields that

$$c_i := \text{rk}(C_i) = \text{rk}(Z_i) + \text{rk}(B_{i-1})$$

and

$$\text{rk}(Z_i) = \text{rk}(B_i) + \text{rk}(H_i).$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over $i$ to get that $\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \text{rk}(H_i)$.

Finally, we apply this result to the cellular chain complex $C_i = H_i(X_i, X_{i-1})$, and use the identification between cellular and singular homology. □
Example 0.1.3. If \( M_g \) and \( N_g \) denote the orientable and resp. nonorientable closed surfaces of genus \( g \), then \( \chi(M_g) = 1 - 2g + 1 = 2(1 - g) \) and \( \chi(N_g) = 1 - g + 1 = 2 - g \). So all the orientable and resp. non-orientable surfaces are distinguished from each other by their Euler characteristic, and there are only the relations \( \chi(M_g) = \chi(N_{2g}) \).

### 0.2 Lefschetz Fixed Point Theorem

Let \( G \) be a finitely generated abelian group. Given an endomorphism \( \varphi : G \to G \), define its trace by

\[
\text{Tr}(\varphi) = \text{Tr}(\overline{\varphi} : G/\text{Torsion}(G) \to G/\text{Torsion}(G))
\]

where the latter trace is the linear algebraic trace of the map \( \overline{\varphi} : \mathbb{Z}^r \to \mathbb{Z}^r \).

Definition 0.2.1. If \( X \) has the homotopy type of a finite cellular (or simplicial) complex and \( f : X \to X \) is a continuous map, then the Lefschetz number of \( f \) is defined to be

\[
\tau(f) = \sum_{i=0}^{\dim(X)} (-1)^i \cdot \text{Tr}(f_* : H_i(X) \to H_i(X)).
\]

Remark 0.2.2. Notice that homotopic maps have the same Lefschetz number since they induce the same maps on homology.

Example 0.2.3. If \( f \simeq \text{id}_X \), then \( \tau(f) = \chi(X) \). This follows from the fact the map induced in homology by the identity map is the identity homomorphism, and the trace of the latter is the corresponding Betti number of \( X \).

Theorem 0.2.4. (Lefschetz)

If \( X \) is a retract of a finite cellular (or simplicial) complex and if the continuous map \( f : X \to X \) satisfies \( \tau(f) \neq 0 \), then \( f \) has a fixed point.

Before proving this theorem, let us consider a few examples.

Example 0.2.5. Suppose that \( X \) has the homology of a point (up to torsion). Then

\[
\tau(f) = \text{Tr}(f_* : H_0(X) \to H_0(X)) = 1.
\]

This follows from the fact that all the other homology groups are zero and that the map induced on \( H_0 \) is the identity.

This example leads immediately to two nontrivial results, the first of which is the Brower fixed point theorem.

Example 0.2.6. (Brower) If \( f : D^n \to D^n \) is continuous then \( f \) has a fixed point.

Example 0.2.7. If \( X = \mathbb{RP}^{2n} \), then modulo torsion \( X \) has the homology of a point. Therefore any continuous map \( f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n} \) has a fixed point.
Finally we are led to an example which does not follow from the computation for a point.

**Example 0.2.8.** If \( f : S^n \to S^n \) is a continuous map and \( \deg(f) \neq (-1)^{n+1} \), then \( f \) has a fixed point. To verify this, we compute

\[
\tau(f) = \text{Tr}(f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \cdot \text{Tr}(f_* : H_n(S^n) \to H_n(S^n))
\]

\[
= 1 + (-1)^n \cdot \deg(f)
\]

\[
\neq 0.
\]

**Corollary 0.2.9.** If \( a : S^n \to S^n \) is the antipodal map, then \( \deg(a) = (-1)^{n+1} \).

Now we return to outlining the proof. Since every finite CW complex is homotopy equivalent to a simplicial complex of the same dimension, it suffices to assume that the space \( X \) is a simplicial complex.

**Definition 0.2.10.** If \( K \) and \( L \) are simplicial complexes and \( f : K \to L \) is a linear map which sends each simplex of \( K \) to a simplex in \( L \) so that vertices map to vertices, then \( f \) is said to be simplicial.

Note that a simplicial map is uniquely determined by its values on vertices. The *simplicial approximation theorem* asserts that given any map \( f \) from a finite simplicial complex to an arbitrary simplicial complex, we can find a map \( g \) in the homotopy class of \( f \) so that \( g \) is simplicial in the above sense with respect to some finite iteration of barycentric subdivisions of the domain.

**Theorem 0.2.11.** If \( K \) is a finite simplicial complex and \( L \) is an arbitrary simplicial complex, then for any map \( f : K \to L \) there is a map in the homotopy class of \( f \) which is simplicial with respect to some iterated barycentric subdivision of \( K \).

The proof of this result is omitted. We now proceed to the Lefschetz theorem.

**Proof.** (sketch)

Let us suppose that \( f \) has no fixed points. The general case reduces to the case when \( X \) is a finite simplicial complex. Indeed, if \( r : K \to X \) is a retraction of a finite simplicial complex \( K \) onto \( X \), the composition \( f \circ r : K \to X \subset K \) has exactly the same fixed points as \( f \) and since \( r_* : H_i(K) \to H_i(X) \) is projection onto a direct summand, we have that \( \text{Tr}(f_* \circ r_*) = \text{Tr}(f_*) \), so \( \tau(f \circ r) = \tau(f) \). We therefore take \( X \) to be a finite simplicial complex.

\( X \) is compact and there exists a metric \( d \) on \( X \) so that \( d \) restricts to the Euclidean metric on each simplex of \( X \); choose such a metric. If \( f \) has no fixed points, we can find a uniform \( \epsilon \) for which \( d(x, f(x)) > \epsilon \) by the standard covering trick. Via repeated barycentric subdivision of \( X \) we can construct \( L \) so that for each vertex, the union of all simplices containing that vertex has diameter less than \( \frac{\epsilon}{2} \). Applying the simplicial approximation theorem we can find a subdivision \( K \) of \( L \) and a simplicial map \( g : K \to L \) so that \( g \) lies in the homotopy class of \( f \). Moreover, we may take \( g \) so that \( f(\sigma) \) lies in the subcomplex of \( X \) consisting of all simplices containing \( \sigma \). Again, by repeated barycentric subdivision we may choose \( K \) so
that each simplex in $K$ has diameter less than $\frac{\epsilon}{2}$. In particular then $g(\sigma) \cap \sigma = \emptyset$ for each $\sigma \in K$. Notice $\tau(g) = \tau(f)$ since $f$ and $g$ are homotopic.

Since $g$ is simplicial, $K_n$ maps to $L_n$ (that is, $g$ sends $n$-skeletons to $n$-skeletons). We constructed $K$ as a subdivision of $L$ so that $g(K_n) \subset K_n$ for each $n$.

We will use the algebraic fact that trace is additive for short exact sequences to show that we can replace $H_i(X)$ with $H_i(K_i, K_{i-1})$ in our computation of the Lefschetz number. By essentially the same argument as was used above in the computation of the Euler characteristic and using this fact we obtain that

$$
\tau(g) = \sum_i (-1)^i \cdot \text{Tr}(g_*: H_i(K_i, K_{i-1}) \to H_i(K_i, K_{i-1}))
$$

We have a natural basis for $H_i(K_i, K_{i-1})$ coming from the simplicies $\sigma^i$ in $K_i$. But since $g(\sigma) \cap \sigma = \emptyset$ it follows that $\text{Tr}(g_*: H_i(K_i, K_{i-1}) \to H_i(K_i, K_{i-1})) = 0$ for each $i$. So $\tau(f) = \tau(g) = 0$.

Without appealing to the fact stated before Definition 0.2.10, the cellular case can proved similarly, by using instead a corresponding cellular approximation theorem.

\[\square\]