1 October 7

Definition 1.1. A map \( p : E \to B \) is called a covering if

1. \( p \) is continuous and onto.
2. For all \( b \in B \), there exists an open neighborhood \( U \) of \( b \) which is evenly covered, i.e., \( p^{-1}(U) = \sqcup \alpha V_\alpha \), where the \( v_\alpha \) are disjoint and open, and \( p|_{V_\alpha} : V_\alpha \to U \) is a homeomorphism.

Example 1.2. 1. \( p : \mathbb{R} \to S^1, t \mapsto e^{2\pi it} \)
2. \( id_X : X \to X \)
3. \( p : X \times \{1, \ldots, n\} \to X, (x, k) \mapsto x \)
4. \( p : S^1 \to S^1, z \mapsto z^n \)
5. \( p : S^n \to \mathbb{R}P^n, x \mapsto [x] = \{ \pm x \} \)
6. \( p : \mathbb{C} \to \mathbb{C}^*, z \mapsto e^z \)

7. Products of covering maps: If \( p_i : E_i \to B_i, i = 1, 2 \) are coverings, then \( p_1 \times p_2 : E_1 \times E_2 \to B_1 \times B_2 \) is a covering.

Remark 1.3. 1. Being a covering implies the map is open and locally a homeomorphism.
2. Not any local homeomorphism is a covering.
3. \( p^{-1}(b) \) is discrete, for all \( b \in B \) (by disjointness.)

Definition 1.4. Let \( p_1 : E_1 \to B, p_2 : E_2 \to B \) be two coverings. We say \( p_1 \) and \( p_2 \) are equivalent if there exists a homeomorphism \( f : E_1 \to E_2 \) such that \( p_2 \circ f = p_1 \). Note: This is an equivalence relation.

Problem: Find all coverings of a space \( B \) (up to equivalence.)
Lemma 1.5. If \( p: E \to B \) is a covering, \( B_0 \subset B \), and \( E_0 := p^{-1}(B_0) \), then \( p|_{E_0}: E_0 \to B_0 \) is a covering.

Example 1.6. Let \( p: \mathbb{R}^2 \to T^2 \) be a covering. Overlay the integer lattice on \( \mathbb{R}^2 \), and identity each square with a torus in the usual way. Let \( p_0 = (1,0) \in S^1 \), and let \( B_0 = S^1 \times \{p_0\} \cup \{p_0\} \times S^1 \). Then \( p^{-1}(B_0) = \mathbb{R} \times \mathbb{Z} \cup \zeta \times \mathbb{R} \) (so it gets rid of the inside of the squares.)

Theorem 1.7 (Path lifting property). Let \( P: E \to B \) be a covering, \( b_0 \in B \), and \( e_0 \in p^{-1}(b_0) \). If \( \gamma: I \to B \) is a path in \( B \) starting at \( b_0 \), then there is a unique lift \( \tilde{\gamma}_{e_0}: I \to E \) such that \( \tilde{\gamma}_{e_0}(0) = e_0 \).

The proof of this theorem follows from the previous lemma.

Theorem 1.8 (Homotopy lifting property). Let \( F: I \times I \to B \) be a homotopy with \( b_0 := F(0,0) \). Then there is a unique lift \( \tilde{F}: I \times I \to E \) of \( F \) such that \( \tilde{F}(0,0) = e_0 \).

Corollary 1.9. If \( \gamma_1, \gamma_2 \) are paths in \( B \) which are homotopic by some \( F \), then \( \gamma_1(0) = \gamma_2(0) = b_0 \), then \( (\tilde{\gamma}_1)_{e_0} \sim_{\tilde{F}} (\tilde{\gamma}_2)_{e_0} \). In particular, these lifts have the same endpoints: \( (\tilde{\gamma}_1)_{e_0}(1) = (\tilde{\gamma}_2)_{e_0}(1) \).

Definition 1.10. Let \( b_0 \in B \). For \( e_0 \in p^{-1}(b_0) \), define
\[
\Phi_{e_0}: \pi_1(B, b_0) \to p^{-1}(b_0)
\]
\[
[\gamma] \mapsto \tilde{\gamma}_{e_0}(1)
\]

Note that by corollary \ref{cor1.9} this map is well-defined.

Theorem 1.11. Let \( \Phi_{e_0} \) be defined as above. Then \( \Phi_{e_0} \) is onto if \( E \) is path-connected, and it is injective if \( E \) is simply connected.

Proof. First suppose that \( E \) is path-connected. Let \( e_1 \in p^{-1}(b_0) \), and let \( \delta \) be a path in \( E \) from \( e_0 \) to \( e_1 \). Then \( p \circ \delta \) is a path in \( B \). Further, \( \gamma := p \circ \delta: I \to B \) is a loop in \( B \) with base point \( b_0 \). Then \( \delta \) is a lift of \( \gamma \) starting at \( e_0 \). Then we have \( \phi_{e_0}([\gamma]) = \tilde{\gamma}_{e_0}(1) = \delta(1) = \delta_1 \), so \( \phi_{e_0} \) is surjective. Note that the equality \( \tilde{\gamma}_{e_0}(1) = \delta(1) \) comes from the uniqueness of lifts.

Now suppose \( E \) is simply connected. Let \( \gamma_1, \gamma_2 \) be loops in \( B \) with base point \( b_0 \) such that \( \phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2]) = e_1 \). By definition, this means that \( (\tilde{\gamma}_1)_{e_0}(1) = (\tilde{\gamma}_2)_{e_0}(1) \). To show that \( \phi_{e_0} \) is injective, we must show \( \gamma_1 \sim \gamma_2 \). Since \( E \) is simply connected, there is a unique homotopy class of paths from \( e_0 \) to \( e_1 \), so \( (\tilde{\gamma}_1)_{e_0} \sim (\tilde{\gamma}_2)_{e_0} \) by some homotopy \( F \). This gives a homotopy \( p \circ F: I \times I \to B \) from \( p \circ (\tilde{\gamma}_1)_{e_0} = \gamma_1 \) to \( p \circ (\tilde{\gamma}_2)_{e_0} = \gamma_2 \), which shows that \( \phi_{e_0} \) is injective.

Example 1.12. Let \( p: S^n \to \mathbb{RP}^n \) be a covering. For \( n \geq 2 \), \( S^n \) is path-connected and simply connected. Then by theorem \ref{thm1.11} \( \Phi_{e_0}: \pi_1(\mathbb{RP}^n, b_0) \to p^{-1}(b_0) \) is a bijection. Since \( \# p^{-1}(b_0) = 2 \), it must be that \( \pi_1(\mathbb{RP}^n, b_0) \cong \mathbb{Z}/2\mathbb{Z} \).
Example 1.13. Let \( p: \mathbb{R} \to S^1 \), \( t \mapsto e^{2\pi it} \). Since \( \mathbb{R} \) is both simply connected and path-connected, theorem [1.11] tells us that \( \phi_e: \pi_1(S^1, b_0) \to \mathbb{Z} \) is a bijection. To show that the groups are isomorphic, we now need to show that \( \phi_e \) is a homomorphism. Let \( \gamma, \delta \in \pi_1(S^1, b_0) \), and let \( \tilde{\gamma}_0, \tilde{\delta}_0 \) be their lifts in \( \mathbb{R} \). Let \( \tilde{\gamma}_0(1) = n \in \mathbb{Z}, \tilde{\delta}_0(1) = m \in \mathbb{Z} \). By definition, \( \phi_e([g]) = n, \phi_e([\delta]) = m \).

Claim 1.14. \( \phi_e([g] \cdot [\delta]) = n + m \) (i.e., it is a homomorphism.)

Proof. We have

\[ \phi_e([\delta] \cdot [\gamma]) = \phi_e([\delta \ast \gamma]) = (\tilde{\gamma} \ast \tilde{\delta})(1) = (\tilde{\gamma}_0 \ast \tilde{\delta}_0)(1) = \tilde{\delta}^*(1) = n + m \]

Now set \( \tilde{\delta}^*(t) = n + \tilde{\delta}_0(t) \) so that \( \tilde{\delta}^*(0) = n, \tilde{\delta}^*(1) = n + m \). Thus, \( \phi_e \) is a homomorphism and therefore an isomorphism. \( \square \)

Proposition 1.15. If \( p: E \to B \) is a covering and \( B \) is path-connected, and \( b_0, b_1 \in B \), there is a bijection \( p^{-1}(b_0) \to p^{-1}(b_1) \).

Proof. Let \( \gamma \) be a path in \( B \) from \( b_0 \) to \( b_1 \) (which exists because \( B \) is path-connected.) Define the bijection \( f_\gamma: p^{-1}(b_0) \to p^{-1}(b_1) \) by \( e_0 \mapsto \tilde{\gamma}_{e_0}(1) \). It has inverse \( (f_\gamma)^{-1} = f_\gamma \). \( \square \)

2 October 9

Proposition 2.1. Let \( E \) be path connected, \( p: E \to B \) a covering, and \( p(e_0) = b_0 \). Then \( p_*: \pi_1(E, e_0) \to \pi_1(B, b_0) \) is injective. Further, if \( e_0 \) is changed to some other \( e_1 \in p^{-1}(b_0) \), then the images of \( p_* \) are conjugate in \( \pi_1(B, b_0) \).

Proof. Let \( p_*([\gamma_1]) = p_*([\gamma_2]) \). Then \( P \circ \gamma_1 \sim p \circ \gamma_2 \) by some homotopy \( F \). By homotopy lifting, we have that \( (P \circ \gamma_1)_{e_0} \sim (p \circ \gamma_2)_{e_0} \), which implies that \( \gamma_1 \sim \gamma_2 \), by the uniqueness of lifts. Thus, \( p_* \) is injective.

Now let \( e_1 \) be a different point in the fiber of \( p \) over \( b_0 \). Define \( H_1 = P_*\pi_1(E, e_0), H_2 = p_*\pi_1(E, e_1) \). We want to show these are conjugate. First let \( \delta \) be a path in \( E \) from \( e_0 \to e_1 \). Then the following diagram commutes:

\[
\begin{array}{ccc}
\pi_1(E, e_0) & \xrightarrow{p_*} & \pi_1(B, b_0) \\
\downarrow{\delta_\#} & & \downarrow{(p \circ \delta)_\#} \\
\pi_1(E, e_1) & \xrightarrow{} & \pi_1(B, b_0)
\end{array}
\]

Note that \( \delta_\# \) is an isomorphism. So we have \( H_1 \cong (p \circ \delta)_\#H_2 \), by conjugation with \( [p \circ \delta] \). \( \square \)

Theorem 2.2. Let \( E \) be path-connected, \( p: E \to B \) a covering map, and \( e_0 \in p^{-1}(b_0) \). Let \( H := p_*\pi_1(E, e_0) \leq \pi_1(B, b_0) \). Then:
a) A closed path $\gamma$ in $B$ based at $b_0$ lifts to a loop in $E$ at $e_0$ iff $[\gamma] \in H$.

b) $\phi_{e_0}: H \setminus \pi_1(B, b_0) \to p^{-1}(b_0), [\gamma] \mapsto \tilde{\gamma}_{e_0}(1)$ is a bijection. In particular $\#p^{-1}(b_0) = [\pi_1(B, b_0) : p_*\pi_1(E, e_0)]$.

Proof of (b). First show that $\phi_{e_0}$ is well-defined, i.e., if $[\delta] \in H$, then $\phi_{e_0}([\delta \cdot [\gamma]]) = \phi_{e_0}([\gamma])$. We have

$$\phi_{e_0}([\gamma] \cdot [\delta]) = \phi_{e_0}(\delta \ast \gamma) = (\tilde{\delta}_{e_0} \ast \tilde{\gamma}_{e_0}(1))(1) = \tilde{\gamma}_{\delta e_0}(1)$$

By part (a), since $[\delta] \in H$, we have that $\tilde{\delta}_{e_0}(1) = e_0$. Thus, $\tilde{\gamma}_{\delta e_0}(1) = \phi_{e_0}([\gamma])$, so it’s well defined. From last class we know that $\phi_{e_0}$ is onto, so it remains to show that it’s injective.

Suppose that $\phi_{e_0}([\gamma_1]) = \phi_{e_0}([\gamma_2])$. By definition, this means that $(\tilde{\gamma}_{1})_{e_0}(1) = (\tilde{\gamma}_{2})_{e_0}(1)$. Thus, $(\tilde{\gamma}_{1})_{e_0} \ast (\tilde{\gamma}_{2})_{e_0}$ is a loop in $E$ based at $e_0$, which is in turn a lift of $\gamma_1 \ast \gamma_2$. By (a), $[\gamma_1 \ast \gamma_2] \in H$. Finally, $[\gamma_1] = [\gamma_1 \ast \gamma_2] = [\gamma_2]$. Note that $[\gamma_1 \ast \gamma_2] \in H$, so $\gamma_1, \gamma_2$ are equivalent in the set of cosets. Thus, the function is injective.

**Theorem 2.3** (Lifting Lemma). Let $E, B, Y$ be path-connected and locally path-connected (i.e., for all $x \in X$ and for all neighborhoods $U_x$ of $x$, there exists a $V_k$ which is path connected-connected, contains $x$, and is contained in $U_x$.) Let $p: E \to B$ be a cover, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, and $f: Y \to B$ such that $f(y_0) = b_0$. Then there exists a lift $\tilde{f}: Y \to E$ such that $\tilde{f}(y_0) = e_0, p \circ \tilde{f} = f$ iff $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$.

![Diagram](https://via.placeholder.com/150)

**Proof.** The $\Rightarrow$ direction is clear. Let $y \in Y$. How should we define $\tilde{f}(y)$? Let $\alpha$ be a path in $Y$ from $y_0$ to $y_1$. Then $f \circ \gamma$ is a path in $B$ starting at $b_0$. Define $\tilde{f}(y) := (f \circ \alpha)(1)$. We have $(p \circ \tilde{f})(y) = p \circ (f \circ \alpha)_{e_0}(1) = f \circ \alpha(1) = f(y)$. Thus, $\tilde{f}$ is actually a lift.

Now we need to show $\tilde{f}$ is well-defined (i.e., independent of $\alpha$). If $\beta$ is another path in $Y$ from $y_0$ to $y$, then $\alpha \ast \beta \in \pi_1(Y, y_0)$, so $f \circ (\alpha \ast \beta) \in f_*\pi_1(Y, y_0)$. Now by assumption, we have $f_*\pi_1(Y, y_0) \subset p_*\pi_1(E, e_0)$. This means that $(f \circ (\alpha \ast \beta))_{e_0}$ is a loop at $e_0$. Then we have

$$(\tilde{f} \circ \alpha)_{e_0} \ast (\tilde{f} \circ \beta)_{e_0} = (\tilde{f} \circ \alpha) \ast (\tilde{f} \circ \beta) = (\tilde{f} \circ \alpha) \ast (\tilde{f} \circ \beta)$$

This means that $(\tilde{f} \circ \alpha)_{e_0} = (\tilde{f} \circ \beta)_{e_0}$, which is what we wanted to show.

Now we need to show that $\tilde{f}$ is continuous. Let $y \in Y$, and let $U$ be a path connected neighborhood of $f^{-1}(y_1) \in B$ (which exists by the locally-connected assumption). Let $V$ be the slice in $p^{-1}(U)$ which contains $\tilde{f}(y_1)$. By the continuity of $f$, there is some path-connected neighborhood of $y$, say $W$, in $Y$ such that $f(W) \subset U$. Then $\tilde{f} = (p|_{V})^{-1} \circ f|_{W}$ is continuous.
Corollary 2.4. If $Y$ is simply connected, then such a lift always exists.

Proposition 2.5. (Lift uniqueness) If $Y$ is connected and $\tilde{f}_1, \tilde{f}_2 : Y \to E$ are two lifts as in the previous theorem, then $\tilde{f}_1 = \tilde{f}_2$.

Proof. Let $A = \{y : \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \emptyset$. We will show $A = Y$ by showing that $A$ is both open and closed. Let $y \in Y$, and let $U$ be an evenly covered neighborhood of $f(y)$ in $B$. Then we have $p^{-1}(u) = \sqcup \tilde{U}_\alpha$ such that $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homeomorphism. Let $\tilde{U}_1, \tilde{U}_2$ be the $\tilde{U}_\alpha$ containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$, respectively. Note that the $\tilde{f}_i$ are continuous, so there is a neighborhood $N$ of $y$ such that $\tilde{f}_1(N) \subset \tilde{U}_1$ and $\tilde{f}_2(N) \subset \tilde{U}_2$. If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$, then $\tilde{U}_1 \neq \tilde{U}_2$, so $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$. This means that $\tilde{f}_1 \neq \tilde{f}_2$ on $N$, so $A$ is closed (as the complement of an open set). On the other hand, if $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $\tilde{U}_1 = \tilde{U}_2$, which implies that $\tilde{f}_1 = \tilde{f}_2$ on $N$ (since $pf_1 = pf_2 = f$, and $p$ is injective on $\tilde{U}_1 = \tilde{U}_2$). Thus, $A$ is open, which proves the proposition. \hfill \Box

Exercise 1. Any continuous map $\mathbb{RP}^2 \to S^1$ is null-homotopic.