1 Homotopy Groups

Definition 1. For $n \geq 0$ and $X$ a topological space with $x_0 \in X$, define

$$\pi_n(X) = \{f : (I^n, \partial I^n) \to (X, x_0)\}/\sim$$

where $\sim$ is the usual homotopy of maps.

Then we have the following diagram of sets:

$$(I^n, \partial I^n) \xrightarrow{f} (X, x_0) \xrightarrow{g} (I^n/\partial I^n, \partial I^n/\partial I^n)$$

Now we have $(I^n/\partial I^n, \partial I^n/\partial I^n) \simeq (S^n, s_0)$. So we can also define

$$\pi_n(X, x_0) = \{g : (S^n, s_0) \to (X, x_0)\}/\sim$$

where $g$ is as above.

Remark 1. If $n = 0$, then $\pi_0(X)$ is the set of connected components of $X$. Indeed, we have $I^0 = \text{pt}$ and $\partial I^0 = \emptyset$, so $\pi_0(X)$ consists of homotopy classes of maps from a point into the space $X$.

Now we will prove several results analogous to the $n = 1$ case.

Proposition 1. If $n \geq 1$, then $\pi_n(X, x_0)$ is a group with respect to the following operation $+$:

$$(f + g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \ldots, s_n) & \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$

(Note that if $n = 1$, this is the usual concatenation of paths/loops.)

Proof. First note that since only the first coordinate is involved in this operation, the same argument used to prove $\pi_1$ is a group is valid here as well. Then the identity element is the constant map taking all of $I^n$ to $x_0$ and inverses are given by $-f = f(1 - s_1, s_2, \ldots, s_n)$. \qed
Proposition 2. If $n \geq 2$, then $\pi_n(X, x_0)$ is abelian.

Intuitively, since the $+$ operation only involves the first coordinate, there is enough space to “slide $f$ past $g$ ”.

Proof. Let $n \geq 2$ and let $f, g \in \pi_n(X, x_0)$. We wish to show $f + g \simeq g + f$. Consider the following figures:

We first shrink the domains of $f$ and $g$ to smaller cubes inside $I^n$ and map the remaining region to the base point $x_0$. Note that this is possible since both $f$ and $g$ map to $x_0$ on the boundaries, so the resulting map is continuous. Then there is enough room to slide $f$ past $g$ inside $I^n$. We then enlarge the domains of $f$ and $g$ back to their original size and get $g + f$. So we have constructed a homotopy between $f + g$ and $g + f$ and hence $\pi_n(X, x_0)$ is abelian. \hfill \qed

If we view $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, then we have the following visual representation of $f + g$ (one can see this by collapsing boundaries in the cube interpretation).

Now recall that if $X$ is path-connected and $x_0, x_1 \in X$, then there is an isomorphism

$$\beta_1 : \pi_1(X, x_0) \to \pi_1(X, x_1)$$
where $\gamma$ is a path from $x_0$ to $x_1$, i.e. we have $\gamma : [0, 1] \to X$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. This $\beta_\gamma$ is given by $\beta_\gamma([f]) = [\gamma^{-1} \cdot f \cdot \gamma]$ for any $[f] \in \pi_1(X, x_0)$.

We claim the same result holds for all $n \geq 1$.

**Proposition 3.** If $n \geq 1$, then there is $\beta_\gamma : \pi_n(X, x_1) \to \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma \cdot \beta]$ where $\gamma \cdot f$ denotes the construction given below and $\gamma$ is a path from $x_1$ to $x_0$.

**Construction:** We shrink the domain of $f$ to a smaller cube inside $I^n$, then insert $\gamma$ radially from $x_1$ to $x_0$ on the boundaries of these cubes.

Then we have the following results:

1. $\gamma \cdot (f + g) \simeq \gamma \cdot f + \gamma \cdot g$
2. $(\gamma \cdot \eta) \cdot f \simeq \gamma \cdot (\eta \cdot f)$ if $\eta$ is another path from $x_0$ to $x_1$
3. $c_{x_0} \cdot f \simeq f$ where $c_{x_0}$ denotes the constant path based at $x_0$.
4. $\beta_\gamma$ is well-defined with respect to homotopies of $\gamma$ or $f$.

Now (1) implies $\beta_\gamma$ is a group homomorphism and (2) and (3) show that $\beta_\gamma$ is invertible. Indeed, if $\pi(t) = \gamma(1 - t)$, then $\beta_{x_0}^{-1} = \beta_{\pi}$. So, as with $n = 1$, if the space $X$ is path-connected, then $\pi_n$ is independent of the choice of base point. Further, if $x_0 = x_1$, then (2) and (3) also imply that $\pi_1(X, x_0)$ acts on $\pi_n(X, x_0)$:

$$\pi_1 \times \pi_n \to \pi_n$$

$$(\gamma, [f]) \mapsto [\gamma \cdot f]$$
**Definition 2.** We say $X$ is an abelian space if $\pi_1$ acts trivially on $\pi_n$ for all $n \geq 1$.

In particular, this means $\pi_1$ is abelian, since the action of $\pi_1$ on $\pi_1$ is by inner-automorphisms, which must all be trivial.

**Exercise 1.** Compute $\pi_3(S^2)$ and $\pi_2(S^2)$.

In general, computing the homotopy groups of spheres is a difficult problem. It turns out that we have $\pi_3(S^2) = \mathbb{Z}$ and $\pi_2(S^2) = \mathbb{Z}$. We will discuss this further at a later point, and for now only remark that since any map $f : S^2 \to S^2$ has an integral degree, we have a map $\pi_2(S^2) \to \mathbb{Z}$ given by $[f] \mapsto \text{deg } f$. Also, the Hopf map $f : S^3 \to S^2$ is an element of $\pi_3(S^2)$.

We next observe that $\pi_n$ is a functor.

**Claim 1.** A map $\phi : X \to Y$ induces group homomorphisms $\phi_* : \pi_n(X,x_0) \to \pi_n(Y,\phi(x_0))$ given by $[f] \mapsto [\phi \circ f]$ for all $n \geq 1$.

**Proof.** From the definition of $+$, it is clear that we have $\phi \circ (f + g) = (\phi \circ f) + (\phi \circ g)$. So $\phi_*([f + g]) = \phi_*([f]) + \phi_*([g])$. Further, if $f \simeq g$, then $\phi \circ f \simeq \phi \circ g$. Indeed, if $\psi_t$ is a homotopy between $f$ and $g$, then $\phi \circ \psi_t$ is a homotopy between $\phi \circ f$ and $\phi \circ g$. Hence $\phi_*$ is a group homomorphism.

**Remark 2.** Induced homomorphisms satisfy the following two properties:

1. $(\phi \circ \psi)_* = \phi_* \circ \psi_*$
2. $(\text{id}_X)_* = \text{id}_{\pi_n(X,x_0)}$

**Corollary 1.** If $\phi : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence, then $\phi_*$ is an isomorphism.

**Proposition 4.** If $p : \tilde{X} \to X$ is a covering map, then $p_* : \pi_n(\tilde{X}, \tilde{x}) \to \pi_n(X, x)$ (where $x = p(\tilde{x})$) is an isomorphism for all $n \geq 2$.

We will delay the proof of this proposition and first consider several examples.

**Example 1.** Consider $\mathbb{R}^n$ (or any contractible space). We have $\pi_i(\mathbb{R}^n) = 0$ for all $i \geq 1$, since $\mathbb{R}^n$ is homotopy equivalent to a point.

**Example 2.** Consider $S^1$. We have the usual covering map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi it}$. We already know $\pi_1(S^1) = \mathbb{Z}$. Now if $n \geq 2$, then we have $\pi_n(S^1) = \pi_n(\mathbb{R}) = 0$.

**Example 3.** Consider $T^n = S^1 \times S^1 \times \cdots \times S^1$ $n$-times. We have $\pi_1(T^n) = \mathbb{Z}^n$. Now there is a covering map $p : \mathbb{R}^n \to T^n$, so we have $\pi_i(T^n) = \pi_i(\mathbb{R}^n) = 0$ for $i \geq 2$. 
Definition 3. If $\pi_n(X) = 0$ for all $n \geq 2$, then $X$ is called aspherical.

We now return to the previous proposition.

Proof. (of Proposition 4) First we claim $p_*$ is surjective. Let $f : (S^n, s_0) \to (X, x)$. Now since $n \geq 2$, we have $\pi_1(S^n) = 0$, so $f_*\pi_1(S^n, s_0) = 0 \subset p_*\pi_1(\tilde{X}, \tilde{x})$ and thus $f$ satisfies the lifting criterion. So there is $\tilde{f} : (S^n, s_0) \to (\tilde{X}, \tilde{x})$ such that $p \circ \tilde{f} = f$. Then $[f] = [p \circ \tilde{f}] = p_*([\tilde{f}])$. Thus $p_*$ is surjective.

Now we claim $p_*$ is injective. Suppose $[\tilde{f}] \in \ker p_*$. So $p_*([\tilde{f}]) = [p \circ \tilde{f}] = 0$. Then $p \circ \tilde{f} \simeq c_x$ via some homotopy $\phi_t : (S^n, s_0) \to (X, x_0)$ with $\phi_1 = f$ and $\phi_0 = c_x$. Let $p \circ \tilde{f} = f$. Again, by the lifting criterion, there is a unique $\tilde{\phi}_t : (S^n, s_0) \to (\tilde{X}, \tilde{x})$ with $p \circ \tilde{\phi}_t = \phi_t$.

Then we have $p \circ \tilde{\phi}_1 = \phi_1 = f$ and $p \circ \tilde{\phi}_0 = c_x$, so by the uniqueness of lifts, we must have $\tilde{\phi}_1 = \tilde{f}$ and $\tilde{\phi}_0 = c_{\tilde{x}}$. Then $\tilde{\phi}_t$ is a homotopy between $\tilde{f}$ and $c_{\tilde{x}}$. So $[\tilde{f}] = 0$. Thus $p_*$ is injective.

\[ \square \]

Proposition 5. Let $\{X_\alpha\}_\alpha$ be a collection of path-connected spaces. Then

\[ \pi_n\left( \prod_\alpha X_\alpha \right) \cong \prod_\alpha \pi_n(X_\alpha) \]

for all $n$.

Proof. First note that a map $f : Y \to \prod_\alpha X_\alpha$ is a collection of maps $f_\alpha : Y \to X_\alpha$. For elements of $\pi_n$, take $Y = S^n$ (note that since all spaces are path-connected, we may drop the reference to base points). For homotopy, take $Y = S^n \times I$.

\[ \square \]

Exercise 2. Find two spaces $X, Y$ such that $\pi_n(X) \cong \pi_n(Y)$ for all $n$ but $X$ and $Y$ are not homotopy equivalent.
Recall that Whitehead’s Theorem states that if a map of CW complexes \( f: X \to Y \) induces isomorphisms on all \( \pi_n \), then \( f \) is a homotopy equivalence. So we must find \( X, Y \) such that there is no continuous map \( f: X \to Y \) inducing the isomorphisms on \( \pi_n \)’s. Consider \( X = S^2 \times \mathbb{RP}^3 \) and \( Y = \mathbb{RP}^2 \times S^3 \). Then \( \pi_n(X) = \pi_n(S^2 \times \mathbb{RP}^3) = \pi_n(S^2) \times \pi_n(\mathbb{RP}^3) \). Now \( S^3 \) is a covering of \( \mathbb{RP}^3 \), so for \( n \geq 2 \), we have \( \pi_n(X) = \pi_1(S^2) \times \pi_n(\mathbb{RP}^3) = \mathbb{Z}_2 \). Similarly, we have \( \pi_n(Y) = \pi_n(\mathbb{RP}^2 \times S^3) = \pi_n(\mathbb{RP}^2) \times \pi_n(S^3) \). Now \( S^2 \) is a covering of \( \mathbb{RP}^2 \), so for \( n \geq 2 \), we have \( \pi_n(Y) = \pi_1(\mathbb{RP}^2) \times \pi_n(S^3) = \mathbb{Z}_2 \). So \( \pi_n(X) = \pi_n(Y) \) for all \( n \). By considering homology groups, however, we see \( X \not\cong Y \). Indeed, we have \( H_5(X) = \mathbb{Z} \) while \( H_5(Y) = 0 \) (since \( \mathbb{RP}^3 \) is oriented while \( \mathbb{RP}^2 \) is not).

\[ \text{Theorem 1. (Hurewicz)} \quad \text{There is a map } \pi_n(X) \to H_n(X) \text{ given by } [f: S^n \to X] \mapsto f_*[S^n] \text{ where } [S^n] \text{ is the fundamental class of } S^n. \text{ Then if } \pi_i(X) = 0 \text{ for all } i < n \text{ where } n \geq 2, \text{ then } H_i(X) = 0 \text{ for } i < n \text{ and } \pi_n(X) \cong H_n(X). \]

\[ \text{Corollary 2. If } X \text{ and } Y \text{ are CW complexes with } \pi_1(X) = \pi_1(Y) = 0 \text{ and } f: X \to Y \text{ induces isomorphisms on all } H_n, \text{ then } f \text{ is a homotopy equivalence.} \]

\[ \text{Exercise 3. Find } X, Y \text{ with the same homology groups, cohomology groups, and cohomology rings, but with different homotopy groups (thus implying } X \not\cong Y). \]

2 Relative Homotopy Groups

Given a triple \((X, A, x_0)\) where \( x_0 \in A \subset X \), we wish to define the relative homotopy groups.

\[ \text{Definition 4. Let } X \text{ be a space and let } A \subset X \text{ and } x_0 \in A. \text{ Let } I^n = \{(s_1, \ldots, s_n) \in I^n : s_n = 0\} \text{ and let } J^{n-1} = \partial I^n \setminus I^{n-1}. \text{ Then define } \]

\[ \pi_n(X, A, x_0) = \{f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)\}/\sim \]

where, as before, \( \sim \) refers to homotopy equivalence.
Alternatively, we can take \( \pi_n(X, A, x_0) = \{ g : (D^n, S^{n-1}, s_0) \to (X, A, x_0) \}/\sim \).

**Proposition 6.** If \( n \geq 2 \), then \( \pi_n(X, A, x_0) \) forms a group under the usual operation. Further, if \( n \geq 3 \), then \( \pi_n(X, A, x_0) \) is abelian.

**Remark 3.** Note that the proposition fails in the case \( n = 1 \). Indeed, we have

\[
\pi_1(X, A, x_0) = \{ f : (I, \{0, 1\}, \{1\}) \to (X, A, x_0) \}/\sim .
\]

Then \( \pi_1(X, A, x_0) \) consists of paths starting anywhere \( A \) and ending at \( x_0 \), so we cannot always concatenate two paths.

As before, a map \( \phi : (X, A, x_0) \to (Y, B, y_0) \) induces homomorphisms \( \phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0) \) for all \( n \).

**Proposition 7.** The relative homotopy groups of \((X, A, x_0)\) fit into a long exact sequence

\[
\cdots \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(A, x_0) \to \pi_0(X, x_0) \to 0
\]

(note that the final terms of the sequence are not groups, just sets) and the map \( \partial \) is defined by

\[
\partial[f] = [f|_{I^{n-1}}]
\]

and all others are induced by inclusions.

Now we wish to describe \( 0 \in \pi_n(X, A, x_0) \). Recall that for \( [f] \in \pi_n(X, x_0) \), we had \( [f] = 0 \) iff \( f \) is homotopic to \( c_{x_0} \). Equivalently, for \( f : (S^n, s_0) \to (X, x_0) \), we have \( [f] = 0 \) iff \( f \) extends to a map \( D^{n+1} \to X \). (Note that the proof is the same as for \( \pi_1 \)).

**Lemma 1.** Let \( [f] \in \pi_n(X, A, x_0) \). Then \( [f] = 0 \) iff \( f \simeq g \) for some map \( g \) with \( \text{im} g \subset A \).
Proof. ($\iff$) Suppose $f \simeq g$ for some $g$ with $\text{im } g \subset A$.

Then we can deform $I^n$ to $J^{n-1}$ as above and so $g \simeq c_{x_0}$. Since homotopy is a transitive relation, we have $f \simeq c_{x_0}$.

($\Rightarrow$) Suppose $[f] = 0$ in $\pi_n(X, A, x_0)$. So $f \simeq c_{x_0}$ via some homotopy $F : I^{n+1} \to X$. Then we may deform $I^n$ inside $I^{n+1}$ (while fixing the boundary) to $J^n$. Composing with $F$, we get a homotopy from $f$ to $g$ with $\text{im } g \subset A$.

Remark 4. The above shows that when proving $[f] = 0$, it is sufficient to show $f \simeq g$ for such a $g$.

We showed that if $X$ is path-connected, then $\pi_1(X)$ acts on $\pi_n(X)$ for all $n$ and $\pi_n(X)$ is independent of our choice of base point. Now we can relax to the following if $x_0, x_1 \in A$ and $\text{im } \gamma \subset A$: 
Lemma 2. If $A$ is path-connected, then $\beta : \pi_n(X, A, x_0) \to \pi_n(X, A, x_1)$ is an isomorphism, where $\gamma$ is a path in $A$ from $x_0$ to $x_1$.

Remark 5. If $x_0 = x_1$, we get an action of $\pi_1(A)$ on $\pi_n(X, A)$.

Definition 5. We say $(X, A)$ is $n$-connected if $\pi_i(X, A) = 0$ for $i \leq n$ and $X$ is $n$-connected if $\pi_i(X) = 0$ for $i \leq n$.

Example 4. $0$-connected $\Rightarrow$ connected and $1$-connected $\Rightarrow$ simply-connected $\Rightarrow$ connected.

3 Homotopy Groups of Spheres

Now we turn our attention to computing $\pi_i(S^n)$. For $i \leq n, i = n + 1, n + 2, n + 3$ and a few more cases, this is known. In general, however, this is a very difficult problem. As noted before, for $i = n$, we should have $\pi_n(S^n) = \mathbb{Z}$ by associating to each map $f : S^n \to S^n$ its degree. For $i < n$, we have $\pi_i(S^n) = 0$. If $f : S^i \to S^n$ is not surjective, i.e. there is $y \in S^n \setminus f(S^i)$, and so $f$ factors through $\mathbb{R}^n$, which is contractible. So we compose $f$ with the retraction $\mathbb{R}^n \to x_0$ and so $f \simeq c_{x_0}$.

BUT there are surjective maps $S^i \to S^n$ for $i < n$, in which case the above proof fails. In this case, we can alter $f$ to make it cellular.

Definition 6. Let $X, Y$ be CW-complexes. Then a map $f : X \to Y$ is called cellular if $f(X^n) \subset Y^n$ for all $n$ (here $X^n$ denotes the $n$-skeleton of $X$).

Theorem 2. Any map between CW-complexes can be homotoped to a cellular map.

Corollary 3. For $i < n$, we have $\pi_i(S^n) = 0$.

Proof. Choose the standard CW-structure on $S^i$ and $S^n$. We may assume $f : S^i \to S^n$ is cellular. Then $f(S^i) \subset (S^n)^i$. But $(S^n)^i$ is a point, so $f$ is a constant map. \qed

Corollary 4. Let $A \subset X$ and suppose all cells of $X \setminus A$ have dimension $> n$. Then $\pi_i(X, A) = 0$ for $i \leq n$.

Proof. Note that there is a similar cellular approximation result for the relative case. Then we may homotope $f : (D^i, S^{i-1}) \to (X, A)$ to $g$ with $g(D^i) \subset X^i$. But $X^i \subset A$, so $\text{im } g \subset A$ and thus $[f] = 0$. \qed

Example 5. $\pi_i(X, X^n) = 0$ for $i \leq n$

Corollary 5. For $i < n$, we have $\pi_i(X) = \pi_i(X^n)$. 
Proof. Consider the long exact sequence of the pair \((X, A)\).

**Theorem 3.** (Suspension Theorem) Let \(f : S^i \to S^n\) and consider its suspension,

\[
\Sigma f : \Sigma S^i = S^{i+1} \to \Sigma S^n = S^{n+1}.
\]

Then from \([f] \in \pi_i(S^n)\), we can create \([\Sigma f] \in \pi_{i+1}(S^{n+1})\). Then this association creates an isomorphism \(\pi_i(S^n) \cong \pi_{i+1}(S^{n+1})\) for \(i < 2n - 1\) and a surjection for \(i = 2n - 1\).

**Corollary 6.** \(\pi_n(S^n)\) is either \(\mathbb{Z}\) or a finite quotient of \(\mathbb{Z}\) (for \(n \geq 2\)), generated by the degree map.

Proof. By the Suspension Theorem, we have the following:

\[
\pi_1(S^1) \rightarrow \pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \cdots
\]

To show \(\pi_1(S^1) \cong \pi_2(S^2)\), we can use the long exact sequence for the homotopy groups of a fibration. (Note: Covering maps are a good example of a fibration for \(F\) discrete).

\[
\begin{tikzcd}
F \ar{r}{f} \ar{d}[swap]{p} & E \ar{d} \ar{r}[swap]{e} & B \\
& \pi_i(F) \rightarrow \pi_i(E) \rightarrow \pi_i(B) \rightarrow \pi_{i-1}(F) \rightarrow \cdots
\end{tikzcd}
\]

Applying the above to the Hopf fibration \(S^1 \hookrightarrow S^3 \xrightarrow{f} S^2\), we have

\[
\cdots \rightarrow \pi_2(S^1) \rightarrow \pi_2(S^3) \rightarrow \pi_2(S^2) \rightarrow \pi_1(S^1) \rightarrow \pi_1(S^3) \rightarrow \cdots
\]

\[
\cdots \pi_3(S^1) \rightarrow 0 \rightarrow \pi_2(S^2) \rightarrow \pi_1(S^1) \rightarrow 0 \rightarrow \cdots
\]

Thus \(\pi_2(S^2) \cong \pi_1(S^1) = \mathbb{Z}\).

**Remark 6.** The above long exact sequence also yields that

\[
\pi_3(S^2) \cong \pi_2(S^2) \cong \mathbb{Z}
\]

**Remark 7.** Unlike the homology and cohomology groups, the homotopy groups of a finite CW-complex can be infinitely generated.
Example 6. For \( n \geq 2 \), consider \( S^1 \vee S^n \). We have \( \pi_n(S^1 \vee S^n) = \pi_n(\tilde{S^1} \vee \tilde{S^n}) \), where \( \tilde{S^1} \vee \tilde{S^n} \) is the universal cover of \( S^1 \vee S^n \), depicted below:

![Diagram of \( S^1 \vee S^n \) with the universal cover](image)

By contracting the segments between integers, we have \( \tilde{S^1} \vee \tilde{S^n} \cong \bigvee_{k \in \mathbb{Z}} S^n_k \). So \( \pi_n(S^1 \vee S^n) = \pi_n(\bigvee_{k \in \mathbb{Z}} S^n_k) \), which is free abelian generated by the inclusions \( S^n_k \hookrightarrow \bigvee_{k \in \mathbb{Z}} S^n_k \).

Note: Several images taken from Hatcher