Math 754
Applications

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1 Fiber bundles

Let $G$ be a topological group (i.e., a topological space endowed with a group structure so that the group multiplication and the inversion map are continuous), acting continuously (on the left) on a topological space $F$. Concretely, such a continuous action is given by a continuous map $\rho: G \times F \to F$, $(g, m) \mapsto g \cdot m$, which satisfies the conditions $(gh) \cdot m = g \cdot (h \cdot m)$ and $e_G \cdot m = m$, for $e_G$ the identity element of $G$.

Any continuous group action $\rho$ induces a map $\text{Ad}_\rho: G \to \text{Homeo}(F)$ given by $g \mapsto (f \mapsto g \cdot f)$, with $g \in G$, $f \in F$. Note that $\text{Ad}_\rho$ is a group homomorphism since $(\text{Ad}_\rho(gh))(f) := (gh) \cdot f = g \cdot (h \cdot f) = \text{Ad}_\rho(g)(\text{Ad}_\rho(h))(f)$. Note that for nice spaces $F$ (such as CW complexes), if we give $\text{Homeo}(F)$ the compact-open topology, then $\text{Ad}_\rho: G \to \text{Homeo}(F)$ is a continuous group homomorphism, and any such continuous group homomorphism $G \to \text{Homeo}(F)$ induces a continuous group action $G \times F \to F$.

We assume from now on that $\rho$ is an effective action, i.e., that $\text{Ad}_\rho$ is injective.

Definition 1.1 (Atlas for a fiber bundle with group $G$ and fiber $F$). Given a continuous map $\pi: E \to B$, an atlas for the structure of a fiber bundle with group $G$ and fiber $F$ on $\pi$ consists of the following data:

a) an open cover $\{U_\alpha\}_\alpha$ of $B$,

b) homeomorphisms (called trivializing charts or local trivializations) $h_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F$ for each $\alpha$ so that the diagram

$$
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\
\pi \downarrow & & \downarrow \text{pr}_1 \\
U_\alpha & \xrightarrow{h_\beta \circ h_\alpha^{-1}} & (U_\alpha \cap U_\beta) \times F
\end{array}
$$

commutes,

c) continuous maps (called transition functions) $g_{\alpha \beta}: U_\alpha \cap U_\beta \to G$ so that the horizontal map in the commutative diagram

$$
\begin{array}{ccc}
\pi^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{h_\alpha} & (U_\alpha \cap U_\beta) \times F \\
\xrightarrow{h_\beta \circ h_\alpha^{-1}} & & (U_\alpha \cap U_\beta) \times F
\end{array}
$$

is given by

$$(x, m) \mapsto (x, g_{\alpha \beta}(x) \cdot m).$$

(By the effectivity of the action, if such maps $g_{\alpha \beta}$ exist, they are unique.)
Definition 1.2. Two atlases $\mathcal{A}$ and $\mathcal{B}$ on $\pi$ are compatible if $\mathcal{A} \cup \mathcal{B}$ is an atlas.

Definition 1.3 (Fiber bundle with group $G$ and fiber $F$). A structure of a fiber bundle with group $G$ and fiber $F$ on $\pi: E \rightarrow B$ is a maximal atlas for $\pi: E \rightarrow B$.

Example 1.4.

1. When $G = \{e_G\}$ is the trivial group, $\pi: E \rightarrow B$ has the structure of a fiber bundle if and only if it is a trivial fiber bundle. Indeed, the local trivializations $h_\alpha$ of the atlas for the fiber bundle have to satisfy $h_\beta \circ h_\alpha^{-1}: (x, m) \mapsto (x, e_G \cdot m) = (x, m)$, which implies $h_\beta \circ h_\alpha^{-1} = \text{id}$, so $h_\beta = h_\alpha$ on $U_\alpha \cap U_\beta$. This allows us to glue all the local trivializations $h_\alpha$ together to obtain a global trivialization $h: \pi^{-1}(B) = E \cong B \times F$.

2. When $F$ is discrete, Homeo($F$) is also discrete, so $G$ is discrete by the effectiveness assumption. So for the atlas of $\pi: E \rightarrow B$ we have $\pi^{-1}(U_\alpha) \cong U_\alpha \times F = \bigcup_{m \in F} U_\alpha \times \{m\}$, so $\pi$ is in this case a covering map.

3. A locally trivial fiber bundle, as introduced in earlier chapters, is just a fiber bundle with structure group Homeo($F$).

Lemma 1.5. The transition functions $g_{\alpha\beta}$ satisfy the following properties:

(a) $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

(b) $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$, for all $x \in U_\alpha \cap U_\beta$.

(c) $g_{\alpha\alpha}(x) = e_G$.

Proof. On $U_\alpha \cap U_\beta \cap U_\gamma$, we have: $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\gamma^{-1}) = h_\alpha \circ h_\gamma^{-1}$. Therefore, since $\text{Ad}_\rho$ is injective (i.e., $\rho$ is effective), we get that $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Note that $(h_\alpha \circ h_\beta^{-1}) \circ (h_\beta \circ h_\alpha^{-1}) = \text{id}$, which translates into

$$(x, g_{\alpha\beta}(x)g_{\beta\alpha}(x) \cdot m) = (x, m).$$

So, by effectiveness, $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = e_G$ for all $x \in U_\alpha \cap U_\beta$, whence $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$.

Take $\gamma = \alpha$ in Property (a) to get $g_{\alpha\beta}(x)g_{\beta\alpha}(x) = g_{\alpha\alpha}(x)$. So by Property (b), we have $g_{\alpha\alpha}(x) = e_G$.

Transition functions determine a fiber bundle in a unique way, in the sense of the following theorem.

Theorem 1.6. Given an open cover $\{U_\alpha\}$ of $B$ and continuous functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow G$ satisfying Properties (a)-(c), there is a unique structure of a fiber bundle over $B$ with group $G$, given fiber $F$, and transition functions $\{g_{\alpha\beta}\}$.
Proof Sketch. Let \( \widetilde{E} = \bigsqcup_{\alpha} U_{\alpha} \times F \times \{ \alpha \} \), and define an equivalence relation \( \sim \) on \( \widetilde{E} \) by
\[
(x, m, \alpha) \sim (x, g_{\alpha\beta}(x) \cdot m, \beta),
\]
for all \( x \in U_{\alpha} \cap U_{\beta} \), and \( m \in F \). Properties (a)-(c) of \( \{ g_{\alpha\beta} \} \) are used to show that \( \sim \) is indeed an equivalence relation on \( \widetilde{E} \). Specifically, symmetry is implied by property (b), reflexivity follows from (c) and transitivity is a consequence of the cycle property (a).

Let \( E = \widetilde{E}/\sim \) be the set of equivalence classes in \( E \), and define \( \pi : E \to B \) locally by \([ (x, m, \alpha) ] \mapsto x\) for \( x \in U_{\alpha} \cap U_{\beta} \), and \( m \in F \). Properties (a)-(c) of \( \{ g_{\alpha\beta} \} \) are used to show that \( \sim \) is indeed an equivalence relation on \( \widetilde{E} \). Specifically, symmetry is implied by property (b), reflexivity follows from (c) and transitivity is a consequence of the cycle property (a).

It remains to show the local triviality of \( \pi \). Let \( p : \widetilde{E} \to E \) be the quotient map, and let \( p_{\alpha} := p|_{U_{\alpha} \times F \times \{ \alpha \}} : U_{\alpha} \times F \times \{ \alpha \} \to \pi^{-1}(U_{\alpha}) \). It is easy to see that \( p_{\alpha} \) is a homeomorphism. We define the local trivializations of \( \pi \) by \( h_{\alpha} := p_{\alpha}^{-1} \).

Example 1.7.

1. Fiber bundles with fiber \( F = \mathbb{R}^n \) and group \( G = GL(n, \mathbb{R}) \) are called rank \( n \) real vector bundles. For example, if \( M \) is a differentiable real \( n \)-manifold, and \( TM \) is the set of all tangent vectors to \( M \), then \( \pi : TM \to M \) is a real vector bundle on \( M \) of rank \( n \).

   More precisely, if \( \varphi_{\alpha} : U_{\alpha} \to \mathbb{R}^n \) are trivializing charts on \( M \), the transition functions for \( TM \) are given by \( g_{\alpha\beta}(x) = d(\varphi_{\alpha} \circ \varphi_{\alpha}^{-1})(\varphi_{\beta}(x)) \).

2. If \( F = \mathbb{R}^n \) and \( G = O(n) \), we get real vector bundles with a Riemannian structure.

3. Similarly, one can take \( F = \mathbb{C}^n \) and \( G = GL(n, \mathbb{C}) \) to get rank \( n \) complex vector bundles.

   For example, if \( M \) is a complex manifold, the tangent bundle \( TM \) is a complex vector bundle.

4. If \( F = \mathbb{C}^n \) and \( G = U(n) \), we get real vector bundles with a hermitian structure.

We also mention here the following fact:

Theorem 1.8. A fiber bundle has the homotopy lifting property with respect to all CW complexes (i.e., it is a Serre fibration). Moreover, fiber bundles over paracompact spaces are fibrations.

Definition 1.9 (Bundle homomorphism). Fix a topological group \( G \) acting effectively on a space \( F \). A homomorphism between bundles \( E' \xrightarrow{\pi'} B' \) and \( E \xrightarrow{\pi} B \) with group \( G \) and fiber \( F \) is a pair \( (f, \hat{f}) \) of continuous maps, with \( f : B' \to B \) and \( \hat{f} : E' \to E \), such that:

1. the diagram

\[
\begin{array}{ccc}
E' & \xrightarrow{\hat{f}} & E \\
\downarrow{\pi'} & & \downarrow{\pi} \\
B' & \xrightarrow{f} & B
\end{array}
\]

commutes, i.e., \( \pi \circ \hat{f} = f \circ \pi' \).
2. if \( \{(U_\alpha, h_\alpha)\}_\alpha \) is a trivializing atlas of \( \pi \) and \( \{(V_\beta, H_\beta)\}_\beta \) is a trivializing atlas of \( \pi' \), then the following diagram commutes:

\[
(V_\beta \cap f^{-1}(U_\alpha)) \times F \xrightarrow{H_\beta} \pi'^{-1}(V_\beta \cap f^{-1}(U_\alpha)) \xrightarrow{f} \pi^{-1}(U_\alpha) \xrightarrow{h_\alpha} U_\alpha \times F
\]

and there exist functions \( d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G \) such that for \( x \in V_\beta \cap f^{-1}(U_\alpha) \) and \( m \in F \) we have:

\[
h_\alpha \circ \hat{f} \circ H_\beta^{-1}(x, m) = (f(x), d_{\alpha\beta}(x) \cdot m).
\]

An isomorphism of fiber bundles is a bundle homomorphism \((f, \hat{f})\) which admits a map \((g, \hat{g})\) in the reverse direction so that both composites are the identity.

**Remark 1.10.** Gauge transformations of a bundle \( \pi : E \to B \) are bundle maps from \( \pi \) to itself over the identity of the base, i.e., corresponding to continuous map \( g : E \to E \) so that \( \pi \circ g = \pi \). By definition, such \( g \) restricts to an isomorphism given by the action of an element of the structure group on each fiber. The set of all gauge transformations forms a group.

**Proposition 1.11.** Given functions \( d_{\alpha\beta} : V_\beta \cap f^{-1}(U_\alpha) \to G \) and \( d_{\alpha'\beta'} : V_{\beta'} \cap f^{-1}(U_{\alpha'}) \to G \) as in (2) above for different trivializing charts of \( \pi \) and \( \pi' \), then for any \( x \in V_\beta \cap V_{\beta'} \cap f^{-1}(U_\alpha \cap U_{\alpha'}) \neq \emptyset \), we have

\[
d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(f(x)) d_{\alpha\beta}(x) g_{\beta\beta'}(x) \tag{1.1}
\]

in \( G \), where \( g_{\alpha'\alpha} \) are transition functions for \( \pi \) and \( g_{\beta\beta'} \) are transition functions for \( \pi' \).

**Proof.** Exercise. \( \square \)

The functions \( \{d_{\alpha\beta}\} \) determine bundle maps in the following sense:

**Theorem 1.12.** Given a map \( f : B' \to B \) and bundles \( E \xrightarrow{\pi} B, E' \xrightarrow{\pi'} B' \), a map of bundles \((f, \hat{f}) : \pi' \to \pi\) exists if and only if there exist continuous maps \( \{d_{\alpha\beta}\} \) as above, satisfying (1.1).

**Proof.** Exercise. \( \square \)

**Theorem 1.13.** Every bundle map \( \hat{f} \) over \( f = \text{id}_B \) is an isomorphism. In particular, gauge transformations are automorphisms.

**Proof Sketch.** Let \( d_{\alpha\beta} : V_\beta \cap U_\alpha \to G \) be the maps given by the bundle map \( \hat{f} : E' \to E \). So, if \( d_{\alpha'\beta'} : V_{\beta'} \cap U_{\alpha'} \to G \) is given by a different choice of trivializing charts, then (1.1) holds on \( V_\beta \cap V_{\beta'} \cap U_\alpha \cap U_{\alpha'} \neq \emptyset \), i.e.,

\[
d_{\alpha'\beta'}(x) = g_{\alpha'\alpha}(x) d_{\alpha\beta}(x) g_{\beta\beta'}(x) \tag{1.2}
\]
in $G$, where $g_{\alpha'\alpha}$ are transition functions for $\pi$ and $g_{\beta'\beta}$ are transition functions for $\pi'$. Let us now invert (1.2) in $G$, and set

$$d_{\beta\alpha}(x) = d^{-1}_{\alpha\beta}(x)$$

to get:

$$d_{\beta\alpha'}(x) = g_{\beta'\beta}(x) d_{\beta\alpha}(x) g_{\alpha'\alpha}(x).$$

So $\{d_{\beta\alpha}\}$ are as in Definition 1.9 and satisfy (1.1). Theorem 1.12 implies that there exists a bundle map $\hat{g} : E \to E'$ over $id_B$.

We claim that $\hat{g}$ is the inverse $\hat{f}^{-1}$ of $\hat{f}$, and this can be checked locally as follows:

$$(x, m) \mapsto (x, d_{\alpha\beta}(x) \cdot m) \mapsto (x, d_{\beta\alpha}(x) \cdot (d_{\alpha\beta}(x) \cdot m)) = (x, d_{\beta\alpha}(x) d_{\alpha\beta}(x) \cdot m) = (x, m).$$

So $\hat{g} \circ \hat{f} = id_{E'}$. Similarly, $\hat{f} \circ \hat{g} = id_E$.

One way in which fiber bundle homomorphisms arise is from the pullback (or the induced bundle) construction.

**Definition 1.14 (Induced Bundle).** Given a bundle $E \xrightarrow{\pi} B$ with group $G$ and fiber $F$, and a continuous map $f : X \to B$, we define

$$f^*E := \{(x, e) \in X \times E \mid f(x) = \pi(e)\},$$

with projections $f^*\pi : f^*E \to X$, $(x, e) \mapsto x$, and $\hat{f} : f^*E \to E$, $(x, e) \mapsto e$, so that the following diagram commutes:

$$\begin{array}{ccc}
f^*E & \xrightarrow{f^*\pi} & E \\
\downarrow f^* & & \downarrow \pi \\
X & \xrightarrow{f} & B \\
\downarrow x & & \downarrow f(x) \\
\end{array}$$

$f^*\pi$ is called the induced bundle under $f$ or the pullback of $\pi$ by $f$, and as we show below it comes equipped with a bundle map $(f, \hat{f}) : f^*\pi \to \pi$.

The above definition is justified by the following result:

**Theorem 1.15.**

(a) $f^*\pi : f^*E \to X$ is a fiber bundle with group $G$ and fiber $F$.

(b) $(f, \hat{f}) : f^*\pi \to \pi$ is a bundle map.
Proof Sketch. Let \( \{(U_\alpha, h_\alpha)\}_\alpha \) be a trivializing atlas of \( \pi \), and consider the following commutative diagram:

\[
\begin{array}{ccc}
(f^*\pi)^{-1}(f^{-1}(U_\alpha)) & \rightarrow & \pi^{-1}(U_\alpha) \xrightarrow{h_\alpha} U_\alpha \times F \\
\downarrow & & \downarrow \\
f^{-1}(U_\alpha) & \xrightarrow{f} & U_\alpha
\end{array}
\]

We have

\[
(f^*\pi)^{-1}(f^{-1}(U_\alpha)) = \{(x, e) \in f^{-1}(U_\alpha) \times \pi^{-1}(U_\alpha) \mid f(x) = \pi(e)\}.
\]

Define

\[ k_\alpha : (f^*\pi)^{-1}(f^{-1}(U_\alpha)) \longrightarrow f^{-1}(U_\alpha) \times F \]

by

\[ (x, e) \mapsto (x, \text{pr}_2(h_\alpha(e))). \]

Then it is easy to check that \( k_\alpha \) is a homeomorphism (with inverse \( k_\alpha^{-1}(x, m) = (x, h_\alpha^{-1}(f(x), m)) \)), and in fact the following assertions hold:

(i) \( \{(f^{-1}(U_\alpha), k_\alpha)\}_\alpha \) is a trivializing atlas of \( f^*\pi \).

(ii) the transition functions of \( f^*\pi \) are \( f^*g_{\alpha\beta} := g_{\alpha\beta} \circ f \), i.e., \( f^*g_{\alpha\beta}(x) = g_{\alpha\beta}(f(x)) \) for any \( x \in f^{-1}(U_\alpha \cap U_\beta) \).

\[ \square \]

Remark 1.16. It is easy to see that \((f \circ g)^*\pi = g^*(f^*\pi)\) and \((id_B)^*\pi = \pi\). Moreover, the pullback of a trivial bundle is a trivial bundle.

As we shall see later on, the following important result holds:

Theorem 1.17. Given a fibre bundle \( \pi : E \to B \) with group \( G \) and fiber \( F \), and two homotopic maps \( f \simeq g : X \to B \), there is an isomorphism \( f^*\pi \simeq g^*\pi \) of bundles over \( X \). (In short, induced bundles under homotopic maps are isomorphic.)

As a consequence, we have:

Corollary 1.18. A fiber bundle over a contractible space \( B \) is trivial.

Proof. Since \( B \) is contractible, \( id_B \) is homotopic to the constant map \( ct \). Let

\[ b := \text{Image}(ct) \xrightarrow{i} B, \]
so \( i \circ c t \simeq \text{id}_B \). We have a diagram of maps and induced bundles:

\[
\begin{array}{ccc}
B & \xrightarrow{\text{id}_B} & B \\
\downarrow & & \downarrow \\
\{b\} & \xrightarrow{i} & B \\
\end{array}
\]

\[
\begin{array}{ccc}
ct^*i^*E & \rightarrow & i^*E \rightarrow E \\
\downarrow & & \downarrow \\
ct^*i^*\pi & \rightarrow & ct^*\pi \rightarrow \pi \\
\end{array}
\]

Theorem 1.17 then yields:

\[ \pi \simeq (\text{id}_B)^*\pi \simeq ct^*i^*\pi. \]

Since any fiber bundle over a point is trivial, we have that \( i^*\pi \simeq \{b\} \times F \) is trivial, hence \( \pi \simeq ct^*i^*\pi \simeq B \times F \) is also trivial.

**Proposition 1.19.** If

\[
\begin{array}{ccc}
E' & \xrightarrow{j} & E \\
\downarrow & & \downarrow \\
B' & \xrightarrow{f} & B \\
\end{array}
\]

is a bundle map, then \( \pi' \simeq f^*\pi \) as bundles over \( B' \).

**Proof.** Define \( h : E' \rightarrow f^*E \) by \( e' \mapsto (\pi'(e'), \tilde{f}(e')) \in B' \times E \). This is well-defined, i.e., \( h(e') \in f^*E \), since \( f(\pi'(e')) = \pi(\tilde{f}(e')) \).

It is easy to check that \( h \) provides the desired bundle isomorphism over \( B' \).

**Example 1.20.** We can now show that the set of isomorphism classes of bundles over \( S^n \) with group \( G \) and fiber \( F \) is isomorphic to \( \pi_{n-1}(G) \). Indeed, let us cover \( S^n \) with two contractible sets \( U_+ \) and \( U_- \) obtained by removing the south, resp., north pole of \( S^n \). Let \( i_{\pm} : U_{\pm} \hookrightarrow S^n \) be the inclusions. Then any bundle \( \pi \) over \( S^n \) is trivial when restricted to \( U_{\pm} \), that is, \( i_{\pm}^*\pi \simeq U_{\pm} \times F \). In particular, \( U_{\pm} \) provides a trivializing cover (atlas) for \( \pi \), and any such bundle \( \pi \) is completely determined by the transition function \( g_{\pm} : U_+ \cap U_- \simeq S^{n-1} \rightarrow G \), i.e., by an element in \( \pi_{n-1}(G) \).

More generally, we aim to “classify” fiber bundles on a given topological space. Let \( B(X, G, F, \rho) \) denote the isomorphism classes (over \( \text{id}_X \)) of fiber bundles on \( X \) with group \( G \)
and fiber $F$, and $G$-action on $F$ given by $\rho$. If $f : X' \to X$ is a continuous map, the pullback construction defines a map

$$f^* : \mathcal{B}(X, G, F, \rho) \to \mathcal{B}(X', G, F, \rho)$$

so that $(id_X)^* = id$ and $(f \circ g)^* = g^* \circ f^*$.

2 Principle Bundles

As we will see later on, the fiber $F$ doesn’t play any essential role in the classification of fiber bundle, and in fact it is enough to understand the set

$$\mathcal{P}(X, G) := \mathcal{B}(X, G, G, m_G)$$

of fiber bundles with group $G$ and fiber $G$, where the action of $G$ on itself is given by the multiplication $m_G$ of $G$. Elements of $\mathcal{P}(X, G)$ are called principal $G$-bundles. Of particular importance in the classification theory of such bundles is the universal principal $G$-bundle $G \hookrightarrow EG \to BG$, with contractible total space $EG$.

Example 2.1. Any regular cover $p : E \to X$ is a principal $G$-bundle, with group $G = \pi_1(X) \setminus \sim p^* \pi_1(E)$. Here $G$ is given the discrete topology. In particular, the universal covering $\tilde{X} \to X$ is a principal $\pi_1(X)$-bundle.

Example 2.2. Any free (right) action of a finite group $G$ on a (Hausdorff) space $E$ gives a regular cover and hence a principal $G$-bundle $E \to E/G$.

More generally, we have the following:

Theorem 2.3. Let $\pi : E \to X$ be a principal $G$-bundle. Then $G$ acts freely and transitively on the right of $E$ so that $E/G \cong X$. In particular, $\pi$ is the quotient (orbit) map.

Proof. We will define the action locally over a trivializing chart for $\pi$. Let $U_\alpha$ be a trivializing open in $X$ with trivializing homeomorphism $h_\alpha : \pi^{-1}(U_\alpha) \sim U_\alpha \times G$. We define a right action on $G$ on $\pi^{-1}(U_\alpha)$ by

$$\pi^{-1}(U_\alpha) \times G \to \pi^{-1}(U_\alpha) \cong U_\alpha \times G$$

$$(e, g) \mapsto e \cdot g := h_\alpha^{-1}(\pi(e), \text{pr}_2(h_\alpha(e)) \cdot g)$$

Let us show that this action can be globalized, i.e., it is independent of the choice of the trivializing open $U_\alpha$. If $(U_\beta, h_\beta)$ is another trivializing chart in $X$ so that $e \in \pi^{-1}(U_\alpha \cap U_\beta)$, we need to show that $e \cdot g = h_\beta^{-1}(\pi(e), \text{pr}_2(h_\beta(e)) \cdot g)$, or equivalently,

$$h_\alpha^{-1}(\pi(e), \text{pr}_2(h_\alpha(e)) \cdot g) = h_\beta^{-1}(\pi(e), \text{pr}_2(h_\beta(e)) \cdot g).$$  \hspace{1cm} (2.1)
After applying $h_\alpha$ and using the transition function $g_{\alpha\beta}$ for $\pi(e) \in U_\alpha \cap U_\beta$, (2.1) becomes

\[(\pi(e), \text{pr}_2(h_\alpha(e)) \cdot g) = h_\alpha h_\beta^{-1} (\pi(e), \text{pr}_2(h_\beta(e)) \cdot g) = (\pi(e), g_{\alpha\beta}(\pi(e)) \cdot (\text{pr}_2(h_\beta(e)) \cdot g)), \tag{2.2}\]

which is guaranteed by the definition of an atlas for $\pi$.

It is easy to check locally that the action is free and transitive. Moreover, $E/G$ is locally given as $U_\alpha \times G/G \cong U_\alpha$, and this local quotient globalizes to $X$. \hfill \Box

The converse of the above theorem holds in some important cases.

**Theorem 2.4.** Let $E$ be a compact Hausdorff space and $G$ a compact Lie group acting freely on $E$. Then the orbit map $E \to E/G$ is a principal $G$-bundle.

**Corollary 2.5.** Let $G$ be a Lie group, and let $H < G$ be a compact subgroup. Then the projection onto the orbit space $\pi : G \to G/H$ is a principal $H$-bundle.

Let us now fix a $G$-space $F$. We define a map

$$\mathcal{P}(X, G) \to \mathcal{B}(X, G, F, \rho)$$

as follows. Start with a principal $G$ bundle $\pi : E \to X$, and recall from the previous theorem that $G$ acts freely on the right on $E$. Since $G$ acts on the left on $F$, we have a left $G$-action on $E \times F$ given by:

$$g \cdot (e, f) \mapsto (e \cdot g^{-1}, g \cdot f).$$

Let

$$E \times_G F := E \times F/G$$

be the corresponding orbit space, with projection map $\omega : E \times_G F \to E/G \cong X$ fitting into a commutative diagram

\[
\begin{array}{ccc}
E \times F & \xrightarrow{\text{pr}_1} & E \\
\downarrow & & \downarrow \omega \\
E & \xrightarrow{\pi} & E \times F/G
\end{array}
\tag{2.3}
\]

**Definition 2.6.** The projection $\omega := \pi \times_G F : E \times_G F \to X$ is called the associated bundle with fiber $F$.

The terminology in the above definition is justified by the following result.

**Theorem 2.7.** $\omega : E \times_G F \to X$ is a fiber bundle with group $G$, fiber $F$, and having the same transition functions as $\pi$. Moreover, the assignment $\pi \mapsto \omega := \pi \times_G F$ defines a one-to-one correspondence $\mathcal{P}(X, G) \to \mathcal{B}(X, G, F, \rho)$.  

Proof. Let \( h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \) be a trivializing chart for \( \pi \). Recall that for \( e \in \pi^{-1}(U_\alpha) \), \( f \in F \) and \( g \in G \), if we set \( h_\alpha(e) = (\pi(e), h) \in U_\alpha \times G \), then \( G \) acts on the right on \( \pi^{-1}(U_\alpha) \) by acting on the right on \( h = pr_2(h_\alpha(e)) \). Then we have by the diagram (2.3) that

\[
\omega^{-1}(U_\alpha) \cong \pi^{-1}(U_\alpha) \times F/(e, f) \sim (e \cdot g^{-1}, g \cdot f)
\]

\[
\cong U_\alpha \times G \times F/(u, h, f) \sim (u, hg^{-1}, g \cdot f).
\]

Let us define

\[
k_\alpha : \omega^{-1}(U_\alpha) \to U_\alpha \times F
\]

by

\[
[(u, h, f)] \mapsto (u, h \cdot f).
\]

This is a well-defined map since \([(u, hg^{-1}, g \cdot f)] \mapsto (u, hg^{-1}g \cdot f) = (u, h \cdot f)\). It is easy to check that \( k_\alpha \) is a trivializing chart for \( \omega \) with inverse induced by \( U_\alpha \times F \to U_\alpha \times G \times F, (u, f) \mapsto (u, id_G, f) \). It is clear that \( \omega \) and \( \pi \) have the same transition functions as they have the same trivializing opens. \( \square \)

The associated bundle construction is easily seen to be functorial in the following sense.

**Proposition 2.8.** If

\[
\begin{array}{ccc}
E' & \xrightarrow{\hat{f}} & E \\
\pi' & \downarrow & \pi \\
X' & \xrightarrow{f} & X
\end{array}
\]

is a map of principal \( G \)-bundles (so \( \hat{f} \) is a \( G \)-equivariant map, i.e., \( \hat{f}(e \cdot g) = \hat{f}(e) \cdot g \)), then there is an induced map of associated bundles with fiber \( F \),

\[
\begin{array}{ccc}
E' \times_G F & \xrightarrow{\hat{f} \times_G id_F} & E \times_G F \\
\pi' & \downarrow & \pi \\
X' & \xrightarrow{f} & X
\end{array}
\]

**Example 2.9.** Let \( \pi : S^1 \to S^1, z \mapsto z^2 \) be regarded as a principal \( \mathbb{Z}/2 \)-bundle, and let \( F = [-1, 1] \). Let \( \mathbb{Z}/2 = \{1, -1\} \) act on \( F \) by multiplication. Then the bundle associated to \( \pi \) with fiber \( F = [-1, 1] \) is the Möbius strip \( S^1 \times_{\mathbb{Z}/2} [-1, 1] = S^1 \times [-1, 1]/(x, t) \sim (a(x), -t) \), with \( a : S^1 \to S^1 \) denoting the antipodal map. Similarly, the bundle associated to \( \pi \) with fiber \( F = S^1 \) is the Klein bottle.

Let us now get back to proving the following important result.

**Theorem 2.10.** Let \( \pi : E \to Y \) be a fiber bundle with group \( G \) and fiber \( F \), and let \( f \simeq g : X \to Y \) be two homotopic maps. Then \( f^* \pi \cong g^* \pi \) over \( id_X \).
It is of course enough to prove the theorem in the case of principal $G$-bundles. The idea of proof is to construct a bundle map over $id_X$ between $f^*\pi$ and $g^*\pi$:

\[
\begin{array}{ccc}
f^*E & \rightarrow & g^*E \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
\]

So we first need to understand maps of principal $G$-bundles, i.e., to solve the following problem: given two principal $G$-bundles bundles $E_1 \overset{\pi_1}{\rightarrow} X$ and $E_2 \overset{\pi_2}{\rightarrow} Y$, describe the set $\mathit{maps}(\pi_1, \pi_2)$ of bundle maps

\[
\begin{array}{ccc}
E_1 & \overset{f}{\rightarrow} & E_2 \\
\downarrow^{\pi_1} & & \downarrow^{\pi_2} \\
X & \overset{f}{\rightarrow} & Y
\end{array}
\]

Since $G$ acts on the right of $E_1$ and $E_2$, we also get an action on the left of $E_2$ by $g \cdot e_2 := e_2 \cdot g^{-1}$. Then we get an associated bundle of $\pi_1$ with fiber $E_2$, namely

\[
\omega := \pi_1 \times_G E_2 : E_1 \times_G E_2 \rightarrow X.
\]

We have the following result:

**Theorem 2.11.** *Bundle maps from $\pi_1$ to $\pi_2$ are in one-to-one correspondence to sections of $\omega$.*

**Proof.** We work locally, so it suffices to consider only trivial bundles.

Given a bundle map $(f, \hat{f}) : \pi_1 \rightarrow \pi_2$, let $U \subset Y$ open, and $V \subset f^{-1}(U)$ open, so that the following diagram commutes (this is the bundle maps in trivializing charts)

\[
\begin{array}{ccc}
V \times G & \overset{\hat{f}}{\rightarrow} & U \times G \\
\downarrow^{\pi_1} & & \downarrow^{\pi_2} \\
V & \overset{f}{\rightarrow} & U
\end{array}
\]

We define a section $\sigma$ in

\[
(V \times G) \times_G (U \times G)
\]

as follows. For $e_1 \in V \times G$, with $x = \pi_1(e_1) \in V$, we set

\[
\sigma(x) = [e_1, \hat{f}(e_1)].
\]
This map is well-defined, since for any $g \in G$ we have:

$$[e_1 \cdot g, \hat{f}(e_1 \cdot g)] = [e_1 \cdot g, \hat{f}(e_1) \cdot g] = [e_1 \cdot g, g^{-1} \cdot \hat{f}(e_1)] = [e_1, \hat{f}(e_1)].$$

Now, it is an exercise in point-set topology (using the local definition of a bundle map) to show that $\sigma$ is continuous.

Conversely, given a section of $E_1 \times_G E_2 \xrightarrow{\omega} X$, we define a bundle by $(f, \hat{f})$ by

$$\hat{f}(e_1) = e_2,$$

where $\sigma(\pi_1(e_1)) = [(e_1, e_2)]$. Note that this is an equivariant map because

$$[e_1 \cdot g, e_2 \cdot g] = [e_1 \cdot g, g^{-1} \cdot e_2] = [e_1, e_2],$$

hence $\hat{f}(e_1 \cdot g) = e_2 \cdot g = \hat{f}(e_1) \cdot g$. Thus $\hat{f}$ descends to a map $f : X \to Y$ on the orbit spaces. We leave it as an exercise to check that $(f, \hat{f})$ is indeed a bundle map, i.e., to show that locally $\hat{f}(v, g) = (f(v), d(v)g)$ with $d(v) \in G$ and $d : V \to G$ a continuous function.

The following result will be needed in the proof of Theorem 2.10.

**Lemma 2.12.** Let $\pi : E \to X \times I$ be a bundle, and let $\pi_0 := i_0^* \pi : E_0 \to X$ be the pullback of $\pi$ under $i_0 : X \to X \times I$, $x \mapsto (x, 0)$. Then $\pi \cong (pr_1)^* \pi_0 \cong \pi_0 \times \text{id}_I$, where $pr_1 : X \times I \to X$ is the projection map.

**Proof.** It suffices to find a bundle map $(pr_1, \hat{pr}_1)$ so that the following diagram commutes

$$\begin{array}{ccc}
E_0 & \xrightarrow{\hat{\pi}_0} & E \\
\downarrow{\pi_0} & & \downarrow{\pi} \\
X & \xleftarrow{i_0} & X \times I \\
& \downarrow{pr_1} & \downarrow{\pi_0} \\
& X & \\
\end{array}$$

By Theorem 2.11, this is equivlant to the existence of a section $\sigma$ of $\omega : E \times_G E_0 \to X \times I$. Note that there exists a section $\sigma_0$ of $\omega_0 : E_0 \times_G E_0 \to X = X \times \{0\}$, corresponding to the bundle map $(\text{id}_X, \text{id}_{E_0}) : \pi_0 \to \pi_0$. Then composing $\sigma_0$ with the top inclusion arrow, we get the following diagram

$$\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\sigma_0} & E \times_G E_0 \\
\uparrow{s} & & \downarrow{\omega} \\
X \times I & \xrightarrow{id} & X \times I \\
\end{array}$$

Since $\omega$ is a fibration, by the homotopy lifting property one can extend $s \sigma_0$ to a section $\sigma$ of $\omega$. \qed

We can now finish the proof of Theorem 2.10.
Proof of Theorem 2.10. Let $H : X \times I \to Y$ be a homotopy between $f$ and $g$, with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Consider the induced bundle $H^*\pi$ over $X \times I$. Then we have the following diagram.

Since $f = H(-, 0)$, we get $f^*\pi = i_0^*H^*\pi$. By Lemma 2.12, $H^*\pi \cong pr_1^*(f^*\pi) \cong pr_1^*(g^*\pi)$, and thus $f^*\pi = i_0^*H^*\pi = i_0^*pr_1^*g^*\pi = g^*\pi$.

We conclude this section with the following important consequence of Theorem 2.11.

**Corollary 2.13.** A principle $G$-bundle $\pi : E \to X$ is trivial if and only if $\pi$ has a section.

**Proof.** The bundle $\pi$ is trivial if and only if $\pi = ct^*\pi'$, with $ct : X \to \text{point}$ the constant map, and $\pi' : G \to \text{point}$ the trivial bundle over a point space. This is equivalent to saying that there is a bundle map

$$
\begin{array}{ccc}
E & \longrightarrow & G \\
\pi & \downarrow & \pi' \\
X & \longrightarrow & \text{point}
\end{array}
$$

or, by Theorem 2.11, to the existence of a section of the bundle $\omega : E \times_G G \to X$. On the other hand, $\omega \cong \pi$, since $E \times_G G \to X$ looks locally like

$$
\pi^{-1}(U_\alpha) \times G \sim U_\alpha \times G \times G / (u, g_1, g_2) \sim (u, g_1g^{-1}, gg_2) \cong U_\alpha \times G,
$$

with the last homeomorphism defined by $[(u, g_1, g_2)] \mapsto (u, g_1g_2)$.

Altogether, $\pi$ is trivial if and only if $\pi : E \to X$ has a section. 

\[ \square \]

### 3 Classification of principal $G$-bundles

Let us assume for now that there exists a principal $G$-bundle $\pi_G : EG \to BG$, with contractible total space $EG$. As we will see below, such a bundle plays an essential role in the classification theory of principal $G$-bundles. Its base space $BG$ turns out to be unique up to homotopy, and it is called the **classifying space for principal $G$-bundles** due to the following fundamental result:
Theorem 3.1. If $X$ is a CW-complex, there exists a bijective correspondence

$$\Phi : \mathcal{P}(X, G) \xrightarrow{\cong} [X, BG]$$

$$f^*\pi_G \leftrightarrow f$$

Proof. By Theorem 2.10, $\Phi$ is well-defined.

Let us next show that $\Phi$ is onto. Let $\pi \in \mathcal{P}(X, G)$, $\pi : E \to X$. We need to show that $\pi \cong f^*\pi_G$ for some map $f : X \to BG$, or equivalently, that there is a bundle map $(f, \tilde{f}) : \pi \to \pi_G$. By Theorem 2.11, this is equivalent to the existence of a section of the bundle $E \times_G EG \to X$ with fiber $EG$. Since $EG$ is contractible, such a section exists by the following:

Lemma 3.2. Let $X$ be a CW complex, and $\pi : E \to X \in \mathcal{B}(X, G, F, \rho)$ with $\pi_i(F) = 0$ for all $i \geq 0$. If $A \subseteq X$ is a subcomplex, then every section of $\pi$ over $A$ extends to a section defined on all of $X$. In particular, $\pi$ has a section. Moreover, any two sections of $\pi$ are homotopic.

Proof. Given a section $\sigma_0 : A \to E$ of $\pi$ over $A$, we extend it to a section $\sigma : X \to E$ of $\pi$ over $X$ by using induction on the dimension of cells in $X - A$. So it suffices to assume that $X$ has the form

$$X = A \cup_{\phi} e^n,$$

where $e^n$ is an $n$-cell in $X - A$, with attaching map $\phi : \partial e^n \to A$. Since $e^n$ is contractible, $\pi$ is trivial over $e^n$, so we have a commutative diagram

$$\begin{array}{ccc}
\pi^{-1}(e^n) & \xrightarrow{\cong} & e^n \times F \\
\downarrow \pi & & \downarrow \text{pr}_1 \\
\partial e^n & \xrightarrow{\sigma} & e^n
\end{array}$$

with $h : \pi^{-1}(e^n) \to e^n \times F$ the trivializing chart for $\pi$ over $e^n$, and $\sigma$ to be defined. After composing with $h$, we regard the restriction of $\sigma_0$ over $\partial e^n$ as given by

$$\sigma_0(x) = (x, \tau_0(x)) \in e^n \times F,$$

with $\tau_0 : \partial e^n \cong S^{n-1} \to F$. Since $\pi_{n-1}(F) = 0$, $\tau_0$ extends to a map $\tau : e^n \to F$ which can be used to extend $\sigma_0$ over $e^n$ by setting

$$\sigma(x) = (x, \tau(x)).$$

After composing with $h^{-1}$, we get the desired extension of $\sigma_0$ over $e^n$.

Let us now assume that $\sigma$ and $\sigma'$ are two sections of $\pi$. To find a homotopy between $\sigma$ and $\sigma'$, it suffices to construct a section $\Sigma$ of $\pi \times id_I : E \times I \to X \times I$. Indeed, if such $\Sigma$ exists, then $\Sigma(x, t) = (\sigma_t(x), t)$, and $\sigma_t$ provides the desired homotopy. Now, by regarding $\sigma$ as a section of $\pi \times id_I$ over $X \times \{0\}$, and $\sigma'$ as a section of $\pi \times id_I$ over $X \times \{1\}$, the question reduces to constructing a section of $\pi \times id_I$, which extends the section over $X \times \{0, 1\}$ defined by $(\sigma, \sigma')$. This can be done as in the first part of the proof. \qed
In order to finish the proof of Theorem 3.1, it remains to show that $\Phi$ is a one-to-one map. If $\pi_0 = f^*\pi_G \cong g^*\pi_G = \pi_1$, we will show that $f \simeq g$. Note that we have the following commutative diagrams:

\[
\begin{array}{ccc}
E_0 = f^*E_G & \xrightarrow{\hat{f}} & E_G \\
\downarrow \pi_0 & & \downarrow \pi_G \\
X = X \times \{0\} & \xrightarrow{f} & B_G \\
E_0 \cong E_1 = g^*E_G & \xrightarrow{\hat{g}} & E_G \\
\downarrow \pi_0 & & \downarrow \pi_G \\
X = X \times \{1\} & \xrightarrow{g} & B_G
\end{array}
\]

where we regard $\hat{g}$ as defined on $E_0$ via the isomorphism $\pi_0 \cong \pi_1$. By putting together the above diagrams, we have a commutative diagram

\[
\begin{array}{ccc}
E_0 \times I & \xleftarrow{\alpha} & E_0 \times \{0, 1\} & \xrightarrow{\hat{\alpha}=(f,0) \cup (g,1)} & E_G \\
\downarrow \pi_0 \times Id & & \downarrow \pi_0 \times \{0, 1\} & & \downarrow \pi_G \\
X \times I & \xleftarrow{\alpha} & X \times \{0, 1\} & \xrightarrow{\alpha=(f,0) \cup (g,1)} & B_G
\end{array}
\]

Therefore, it suffices to extend $(\alpha, \hat{\alpha})$ to a bundle map $(H, \hat{H}) : \pi_0 \times Id \to \pi_G$, and then $H$ will provide the desired homotopy $f \simeq g$.

By Theorem 2.11, such a bundle map $(H, \hat{H})$ corresponds to a section $\sigma$ of the fiber bundle

\[
\omega : (E_0 \times I) \times_G E_0 \to X \times I.
\]

On the other hand, the bundle map $(\alpha, \hat{\alpha})$ already gives a section $\sigma_0$ of the fiber bundle

\[
\omega_0 : (E_0 \times \{0, 1\}) \times_G E_0 \to X \times \{0, 1\},
\]

which under the obvious inclusion $(E_0 \times \{0, 1\}) \times_G E_0 \subseteq (E_0 \times I) \times_G E_0$ can be regarded as a section of $\omega$ over the subcomplex $X \times \{0, 1\}$. Since $EG$ is contractible, Lemma 3.2 allows us to extend $\sigma_0$ to a section $\sigma$ of $\omega$ defined on $X \times I$, as desired.

**Example 3.3.** We give here a more conceptual reasoning for the assertion of Example 1.20. By Theorem 3.1, we have

\[
B(S^n, G, F, \rho) \cong \mathcal{P}(S^n, G) \cong [S^n, BG] = \pi_n(BG) \cong \pi_{n-1}(G),
\]

where the last isomorphism follows from the homotopy long exact sequence for $\pi_G$, since $EG$ is contractible.

Back to the universal principal $G$-bundle, we have the following
Theorem 3.4. Let $G$ be a locally compact topological group. Then a universal principal $G$-bundle $\pi_G : EG \to BG$ exists (i.e., satisfying $\pi_i(EG) = 0$ for all $i \geq 0$), and the construction is functorial in the sense that a continuous group homomorphism $\mu : G \to H$ induces a bundle map $(B\mu, E\mu) : \pi_G \to \pi_H$. Moreover, the classifying space $BG$ is unique up to homotopy.

Proof. To show that $BG$ is unique up to homotopy, let us assume that $\pi'_G : E'_G \to B'_G$ are universal principal $G$-bundles. By regarding $\pi_G$ as the universal principal $G$-bundle for $\pi'_G$, we get a map $f : B'_G \to BG$ such that $\pi'_G = f^*\pi_G$, i.e., a bundle map:

$$
\begin{array}{ccc}
E'_G & \xrightarrow{f} & E_G \\
\downarrow{\pi'_G} & & \downarrow{\pi_G} \\
B'_G & \xrightarrow{f} & BG
\end{array}
$$

Similarly, regarding $\pi'_G$ as the universal principal $G$-bundle for $\pi_G$, there exists a map $g : BG \to B'_G$ such that $\pi_G = g^*\pi'_G$. Therefore,

$$
\pi_G = g^*\pi'_G = g^*f^*\pi_G = (f \circ g)^*\pi_G.
$$

On the other hand, we have $\pi_G = (id_{BG})^*\pi_G$, so by Theorem 3.1 we get that $f \circ g \simeq id_{BG}$. Similarly, we get $g \circ f \simeq id_{B'_G}$, and hence $f : B'_G \to BG$ is a homotopy equivalence.

We will not discuss the existence of the universal bundle here, instead we will indicate the universal $G$-bundle, as needed, in specific examples. \qed

Example 3.5. Recall that we have a fiber bundle

$$
O(n) \xleftarrow{\sim} V_n(\mathbb{R}^\infty) \xrightarrow{\sim} G_n(\mathbb{R}^\infty),
$$

with $V_n(\mathbb{R}^\infty)$ contractible. In particular, the uniqueness part of Theorem 3.4 tells us that $BO(n) \simeq G_n(\mathbb{R}^\infty)$ is the classifying space for rank $n$ real vector bundles. Similarly, there is a fiber bundle

$$
U(n) \xleftarrow{\sim} V_n(\mathbb{C}^\infty) \xrightarrow{\sim} G_n(\mathbb{C}^\infty),
$$

with $V_n(\mathbb{C}^\infty)$ contractible. Therefore, $BU(n) \simeq G_n(\mathbb{C}^\infty)$ is the classifying space for rank $n$ complex vector bundles.

Before moving to the next example, let us mention here without proof the following useful result:

Theorem 3.6. Let $G$ be an abelian group, and let $X$ be a CW complex. There is a natural bijection

$$
T : [X, K(G, n)] \to H^n(X, G)
$$

$$
[f] \mapsto f^*(\alpha)
$$

where $\alpha \in H^n(K(G, n), G) \cong \text{Hom}(H_n(K(G, n), \mathbb{Z}), G)$ is given by the inverse of the Hurewicz isomorphism $G = \pi_n(K(G, n)) \to H_n(K(G, n), \mathbb{Z})$. 17
Example 3.7 (Classification of real line bundles). Let $G = \mathbb{Z}/2$ and consider the principal $\mathbb{Z}/2$-bundle $\mathbb{Z}/2 \hookrightarrow S^\infty \to \mathbb{R}P^\infty$. Since $S^\infty$ is contractible, the uniqueness of the universal bundle yields that $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$. In particular, we see that $\mathbb{R}P^\infty$ classifies the real line (i.e., rank-one) bundles. Since we also have that $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$, we get:

$$\mathcal{P}(X, \mathbb{Z}/2) = [X, B\mathbb{Z}/2] = [X, K(\mathbb{Z}/2, 1)] \cong H^1(X, \mathbb{Z}/2)$$

for any CW complex $X$, where the last identification follows from Theorem 3.6. Let now $\pi$ be a real line bundle on a CW complex $X$, with classifying map $f_\pi : X \to \mathbb{R}P^\infty$. Since $H^*(\mathbb{R}P^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[w]$, with $w$ a generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}/2)$, we get a well-defined degree one cohomology class

$$w_1(\pi) := f^*_\pi(w)$$

called the first Stiefel-Whitney class of $\pi$. The bijection $\mathcal{P}(X, \mathbb{Z}/2) \overset{\cong}{\longrightarrow} H^1(X, \mathbb{Z}/2)$ is then given by $\pi \mapsto w_1(\pi)$, so real line bundles on $X$ are classified by their first Stiefel-Whitney classes.

Example 3.8 (Classification of complex line bundles). Let $G = S^1$ and consider the principal $S^1$-bundle $S^1 \hookrightarrow S^\infty \to \mathbb{C}P^\infty$. Since $S^\infty$ is contractible, the uniqueness of the universal bundle yields that $BS^1 \cong \mathbb{C}P^\infty$. In particular, as $S^1 = GL(1, \mathbb{C})$, we see that $\mathbb{C}P^\infty$ classifies the complex line (i.e., rank-one) bundles. Since we also have that $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, we get:

$$\mathcal{P}(X, S^1) = [X, BS^1] = [X, K(\mathbb{Z}, 2)] \cong H^2(X, \mathbb{Z})$$

for any CW complex $X$, where the last identification follows from Theorem 3.6. Let now $\pi$ be a complex line bundle on a CW complex $X$, with classifying map $f_\pi : X \to \mathbb{C}P^\infty$. Since $H^*(\mathbb{C}P^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$, with $c$ a generator of $H^2(\mathbb{C}P^\infty, \mathbb{Z})$, we get a well-defined degree two cohomology class

$$c_1(\pi) := f^*_\pi(c)$$

called the first Chern class of $\pi$. The bijection $\mathcal{P}(X, S^1) \overset{\cong}{\longrightarrow} H^2(X, \mathbb{Z})$ is then given by $\pi \mapsto c_1(\pi)$, so complex line bundles on $X$ are classified by their first Chern classes.

Remark 3.9. If $X$ is any orientable closed oriented surface, then $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, so Example 3.8 shows that isomorphism classes of complex line bundles on $X$ are in bijective correspondence with the set of integers. On the other hand, if $X$ is a non-orientable closed surface, then $H^2(X, \mathbb{Z}) \cong \mathbb{Z}/2$, so there are only two isomorphism classes of complex line bundles on such a surface.

4 Exercises

1. Let $p : S^2 \to \mathbb{R}P^2$ be the (oriented) double cover of $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is a non-orientable surface, we know by Remark 3.9 that there are only two isomorphism classes of complex line bundles on $\mathbb{R}P^2$: the trivial one, and a non-trivial complex line bundle which we denote by
\[ \pi : E \to \mathbb{R}P^2. \] On the other hand, since \( S^2 \) is a closed orientable surface, the isomorphism classes of complex line bundles on \( S^2 \) are in bijection with \( \mathbb{Z} \). Which integer corresponds to complex line bundle \( p^*\pi : p^*E \to S^2 \) on \( S^2 \)?

2. Consider a locally trivial fiber bundle \( S^2 \hookrightarrow E \overset{\pi}{\to} S^2 \). Recall that such \( \pi \) can be regarded as a fiber bundle with structure group \( G = \text{Homeo}(S^2) \cong SO(3) \). By the classification Theorem 3.1, \( SO(3) \)-bundles over \( S^2 \) correspond to elements in \([S^2, BSO(3)] = \pi_2(BSO(3)) \cong \pi_1(SO(3))\).

(a) Show that \( \pi_1(SO(3)) \cong \mathbb{Z}/2 \). (Hint: Show that \( SO(3) \) is homeomorphic to \( \mathbb{R}P^3 \).)

(b) What is the non-trivial \( SO(3) \)-bundle over \( S^2 \)?

3. Let \( \pi : E \to X \) be a principal \( S^1 \)-bundle over the simply-connected space \( X \). Let \( a \in H^1(S^1, \mathbb{Z}) \) be a generator. Show that

\[ c_1(\pi) = d_2(a), \]

where \( d_2 \) is the differential on the \( E_2 \)-page of the Leray-Serre spectral sequence associated to \( \pi \), i.e., \( E_2^{p,q} = H^p(X, H^q(S^1)) \Rightarrow H^{p+q}(E, \mathbb{Z}) \).

4. By the classification Theorem 3.1, (isomorphism classes of) \( S^1 \)-bundles over \( S^2 \) are given by

\[ [S^2, BS^1] = \pi_2(BS^1) \cong \pi_1(S^1) \cong \mathbb{Z} \]

and this correspondence is realized by the first Chern class, i.e., \( \pi \mapsto c_1(\pi) \).

(a) What is the first Chern class of the Hopf bundle \( S^1 \hookrightarrow S^3 \to S^2 \)?

(b) What is the first Chern class of the sphere (or unit) bundle of the tangent bundle \( TS^2 \)?

(c) Construct explicitly the \( S^1 \)-bundle over \( S^2 \) corresponding to \( n \in \mathbb{Z} \). (Hint: Think of lens spaces, and use the above Exercise 3.)