INTRODUCTORY TOPOLOGY II.

HOMEWORK #2

1. Show that if $X$ is the union of contractible open subsets $A$ and $B$, then all cup products of positive-dimensional classes in $H^*(X)$ are zero. In particular, this is the case if $X$ is a suspension. Conclude that spaces such as $\mathbb{R}P^2$ and $T^2$ cannot be written as unions of two open contractible subsets.

2. Use cup products to compute the map $H^*(\mathbb{C}P^n; \mathbb{Z}) \to H^*(\mathbb{C}P^n; \mathbb{Z})$ induced by the map $\mathbb{C}P^n \to \mathbb{C}P^n$ that is a quotient of the map $\mathbb{C}P^n \to \mathbb{C}P^n$ raising each coordinate to the $d$-th power, $(z_0, \cdots, z_n) \mapsto (z_0^d, \cdots, z_n^d)$, for a fixed integer $d > 0$. (Hint: First do the case $n = 1$.)

3. Use cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \cup S^3$.

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5. Let $\mathbb{H} = \mathbb{R} \cdot 1 + \mathbb{R} \cdot i + \mathbb{R} \cdot j + \mathbb{R} \cdot k$ be the skew-field of quaternions, where $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji, jk = i = -kj, ki = j = -jk$. For a quaternion $q = a + bi + cj + dk, a, b, c, d \in \mathbb{R}$, its conjugate is defined by $\overline{q} = a - bi - cj - dk$. Let $|q| := \sqrt{a^2 + b^2 + c^2 + d^2}$.

   (1) Verify the following formulae in $\mathbb{H}$: $q \cdot \overline{q} = |q|^2, q_1 q_2 = \overline{q_2 q_1}, |q_1 q_2| = |q_1| \cdot |q_2|$. 

   (2) Let $S^7 \subset \mathbb{H} \oplus \mathbb{H}$ be the unit sphere, and let $f : S^7 \to S^4 = \mathbb{H}P^1 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1 q_2^{-1}$. Show that for any $p \in S^4$, the fiber $f^{-1}(p)$ is homeomorphic to $S^3$.

   (3) Let $\mathbb{H}P^n$ be the quaternionic projective space defined exactly as in the complex case as the quotient of $\mathbb{H}^{n+1} \setminus \{0\}$ by the equivalence relation $v \sim \lambda v$, for $\lambda \in \mathbb{H} \setminus \{0\}$. Show that the CW structure of $\mathbb{H}P^n$ consists of only one cell in each dimension $0, 4, 8, \cdots, 4n$, and calculate the homology of $\mathbb{H}P^n$.

   (4) Show that $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$, with $x$ the generator of $H^4(\mathbb{H}P^n; \mathbb{Z})$. 

(5) Show that $S^4 \vee S^8$ and $\mathbb{H}P^2$ are not homotopy equivalent.

6. For a map $f : S^{2n-1} \to S^n$ with $n \geq 2$, let $X_f = S^n \cup_f D^{2n}$ be the CW complex obtained by attaching a $2n$-cell to $S^n$ by the map $f$. Let $a \in H^n(X_f; \mathbb{Z})$ and $b \in H^{2n}(X_f; \mathbb{Z})$ be the generators of respective groups. The Hopf invariant $H(f) \in \mathbb{Z}$ of the map $f$ is defined by the identity $a^2 = H(f)b$.

1. Let $f : S^3 \to S^2 = \mathbb{C} \cup \{\infty\}$ be given by $f(z_1, z_2) = z_1/z_2$, for $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$. Show that $X_f = \mathbb{C}P^2$ and $H(f) = \pm 1$.

2. Let $f : S^7 \to S^4 = \mathbb{H} \cup \{\infty\}$ be given by $f(q_1, q_2) = q_1q_2^{-1}$ in terms of quaternions $(q_1, q_2) \in S^7$, the unit sphere in $\mathbb{H}^2$. Show that $X_f = \mathbb{H}P^2$ and $H(f) = \pm 1$. 