1. (January 2009 Problem 4) Let $V$ be a finite-dimensional vector space over the complex numbers $\mathbb{C}$ and let $T : V \rightarrow V$ be a linear operator on $V$.

(a) If $T$ is diagonalizable on $V$, and if $W$ is a subspace of $V$ with $T(W) \subseteq W$, prove that $T$ is diagonalizable on $W$.

Jordan Canonical Form

The Jordan canonical form, which requires passing to an algebraic extension of the base field containing all the eigenvalues, says that any matrix is conjugate to a block-diagonal matrix whose diagonal entries are $\lambda$-by-$\lambda$ Jordan blocks, which have $\lambda$ on the diagonal, $1$ in the entries directly above the diagonal, and $0$ elsewhere. The Jordan form is unique up to reordering blocks.

Since $T$ is diagonalizable on $V$, the Jordan form of $T$ (on $V$) is diagonal. Thus, the eigenvalues of $T$ each correspond only to ordinary eigenvectors.

The non-ordinary eigenvectors correspond to the Jordan blocks with $1$'s on the super-diagonal. So a matrix is diagonalizable over a field containing all eigenvalues if and only if there are no non-ordinary generalized eigenvectors.

Each eigenvalue of $T|_W$ is also an eigenvalue of $T$. If an eigenvalue $\lambda$ corresponded to a non-ordinary eigenvector of $T|_W$, then $\lambda$ would also correspond to a non-ordinary eigenvector of $T$ in $V$, which is a contradiction. Thus the Jordan form of $T|_W$ is also diagonal, and $T|_W$ is diagonalizable.

(b) If $T$ has the matrix

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
2 & 0 & 0
\end{pmatrix}
$$

with respect to some basis of $V$, decide (with proof) whether $T$ is diagonalizable on $V$.

We have $T(\mathbf{e}_1) = 2\mathbf{e}_1$, $\mathbf{e}_1 \rightarrow \mathbf{e}_1$, $\mathbf{e}_2 \rightarrow \mathbf{e}_2$, $\mathbf{e}_3 \rightarrow \mathbf{0}$. Thus $T^3 \mathbf{e}_1 = \mathbf{0}$, and so $\mathbf{e}_1$ is a non-ordinary eigenvector corresponding to the eigenvalue $\lambda = 0$. Thus $T$ cannot be diagonalizable.

Generalized Eigenspace

Every non-zero matrix $A$ has $n$ linearly independent generalized eigenvectors associated with it. The set spanned by all generalized eigenvectors is the generalized eigenspace of $A$. 
2. Let \( k \) be any field.

(a) Let \( a_1, \ldots, a_d, b_1, \ldots, b_d \) be elements of \( k \). Prove that there is a unique non-zero polynomial \( f \in k[x] \) of degree at most \( d - 1 \) over \( k \) such that \( f(a_i) = b_i \).

Let \( V \) be the vector space of polynomials of degree at most \( d \). Then \( V \) has dimension \( d+1 \) over \( k \), with a basis \( \{1, x, x^2, \ldots, x^d\} \).

Define \( T : V \rightarrow k^{d+1} \) by \( T(f) = (f(a_1), \ldots, f(a_d)) \).

We show that \( T \) is injective, and therefore surjective since it is a map of vector spaces with the same dimension. We have

\[
T(f) = 0 \Rightarrow f(a_i) = 0 \Rightarrow a_i \text{ is a root of } f \text{ for } i = 0, \ldots, d
\]

Thus we have found \( d+1 \) distinct roots of \( f \). Since \( k \) is a field, and since \( f \) has degree at most \( d \), we conclude that \( f = 0 \).

Thus \( T \) is an isomorphism, and so we must have a unique \( f \in k[x] \) of degree at most \( d \) with \( f(a_i) = b_i \).

(b) Is the previous theorem still true if we allow polynomials with more than one variable? Specifically, prove or disprove the following claim: Let \( (a_1, b_1), \ldots, (a_{d+2}, b_{d+2}), c_1, \ldots, c_{d+2} \in k \), then there exists a unique non-zero polynomial \( f \in k[x, y] \) of degree at most \( d \) such that \( f(a_i, b_i) = c_i \).

Let \( d = 1 \), consider the 3 points \((1,0), (0,1), (1,1)\), and let \( c_1 = c_2 = c_3 = 0 \).

If \( f \) has degree at most 1, then \( f(x, y) = a_1x + a_2y + c_3 \) for some \( a_1, a_2, c_3 \).

Suppose \( f(1,0) = f(0,1) = f(1,1) = 0 \). Then we have

\[
\begin{align*}
a_1 + b_1 &= 0, \\
a_2 + b_2 &= 0, \\
a_1 + a_2 + c_3 &= 0.
\end{align*}
\]

This implies \( a_1 = a_2 = b_1 = b_2 = c_3 = 0 \), i.e., \( f = 0 \).
3. (January 2013 Problem 5) Let $W_n$ be the set of $n \times n$ complex matrices $C$ such that the equation 
\[ AB - BA = C \]
has a solution in $n \times n$ matrices $A, B$.

(a) Show that $W_n$ is closed under scalar multiplication and conjugation.

\[ \lambda C = (\lambda A)B - B(\lambda A), \quad PCP^{-1} = (PAP^{-1})(PBP^{-1}) - (PBP^{-1})^*(PAP^{-1})^* \]

(b) Show that the identity matrix is not in $W_n$.

We have $\text{tr}(AB-BA) = \text{tr}(AB) - \text{tr}(BA) = 0$, and $\text{tr}(I) = n \neq 0$.

(c) Give a complete description of $W_2$ (i.e. a criterion for determining whether a matrix $C$ is in $W_2$.)

By part a, we know $C$ is in $W_2$ if and only if its Jordan form is in $W_2$.

By part b, we know every element of $W$ has trace 0. Thus we only need to consider matrices of the form $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ where $b=1$ or $b=0$.

If $a \neq 0$, then $b=0$ since the eigenvalues are distinct. Since $W_2$ is closed under scaling, we only need to consider the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Clearly the zero matrix is in $W_2$. We also have
\[ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

and
\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \].

Thus all matrices with trace zero are in $W_2$, and so $W_2$ consists precisely of these matrices.
4. (January 2014 Problem 2) Let $F$ be a field and $n$ a positive integer. Let $A$ be an $n$ by $n$ matrix over $F$, such that $A^n = 0$ but $A^{n-1} \neq 0$. Show that any $n$ by $n$ matrix $B$ over $F$ that commutes with $A$ is contained in the $F$-linear span of $I, A, \ldots, A^{n-1}$.

Since $A$ is nilpotent, the eigenvalues are all zero. So $A$ can be put in Jordan form over $F$, say $P A P^{-1} = J$.

Now, note that matrices $A, B$ commute if and only if their conjugates $P A P^{-1}$, $P B P^{-1}$ commute. Also, $B$ is in the $F$-linear span of $I, J, I J, \ldots, J^n$. So, without loss of generality, we can assume that $A$ is in Jordan form.

Since the eigenvalues of $A$ are 0, we can write

$$A = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix} = a_1 E_{11} + a_2 E_{22} + \cdots + a_n E_{nn},$$

where $a_i \in \{0, 1\}$.

We first show that $a_i = 1$ for $(i = 1, \ldots, n).$

We have

$$A^k = \left( a_1 E_{11} + a_2 E_{22} + \cdots + a_n E_{nn} \right)^k = a_1 a_2 a_3 \cdots a_n E_{nn},$$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Proceeding inductively, we get

$$A^{n-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Since $A^n \neq 0$, we must have $a_i = 1$ for each $i$, and so $A = E_{11} + E_{22} + \cdots + E_{nn}$.

Now for any matrix $B = \sum b_{ij} E_{ij}$, we have

$$E_{ki} B = \sum b_{kj} E_{kj} \quad \text{and} \quad B E_{ki} = \sum b_{ik} E_{ki}.$$

In other words, $E_{ki}$ moves the $j$th row to the $k$th row (and kills everything else) when acting on the left, and it moves the $k$th column to the $j$th column (and kills everything else) when acting on the right.

This tells us that $A^m = \sum E_{j1i}$ acts on the left by "moving all rows up $m$" (and replacing empty rows with 0), and acts on the right by "moving all columns right $m$".

So a matrix $B$ commutes with $A$ if and only if these right and left actions are equal on $B$, i.e., $B$ must be upper-triangular, and each minor diagonal must have equal values in each entry. We can write

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix}$$

$$= b_{11} I + b_{22} A + \cdots + b_{nn} A^{n-1}.$$
5. (January 2015 Problem 4) Let $F$ be a field, and let $V$ be a finite-dimensional vector space over $F$ that has dimension $n$ ($1 \leq n < \infty$). Let $q$ be a nonzero element of $F$ such that $q^i \neq 1$ for all $1 \leq i \leq n$. Suppose given $F$-linear transformations $X: V \rightarrow V$ and $Y: V \rightarrow V$ such that $XY = qYX$. Show that $XY$ is nilpotent.

Let $\lambda$ be an eigenvalue of $XY$, with $XY \mathbf{v} = \lambda \mathbf{v}$. Then $XY \mathbf{v} = qYX \mathbf{v} = q\lambda \mathbf{v}$, so $q\lambda$ is an eigenvalue of $XY$.

However, we know that $XY$ and $YX$ share the same eigenvalues. Thus we can say that $\lambda, q\lambda, q^2\lambda, \ldots, q^n\lambda$ are all eigenvalues of $XY$. If $\lambda \neq 0$, then these values are distinct, which contradicts the fact that $XY$ can only have $n$ eigenvalues. Thus all eigenvalues of $XY$ are zero, which implies $XY$ is nilpotent.

A matrix is nilpotent if and only if each of its eigenvalues is $0$.

Note: This is only true if you look at all eigenvalues in the algebraic closure of the field. It follows from the fact that the characteristic polynomial must be $X^n$ to have all zero eigenvalues.
1. (August 2014 Problem 3) This problem concerns eigenvectors of linear transformations.

(a) Let $V \neq 0$ be a finite-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Prove that $T$ has an eigenvector.

Since $V$ is finite-dimensional, we can take $T$ to be an $n \times n$ matrix over $\mathbb{C}$.

A vector $v$ is an eigenvector of $T$ if and only if it is in the kernel of the transformation $T-\lambda I$, for some $\lambda$. Thus we only need to show that a $\lambda$ exists so that the kernel of $T-\lambda I$ is nontrivial, which is equivalent to showing $\det(T-\lambda I) = 0$.

Now $\det(T-\lambda I)$ is a $n$th degree polynomial of $\lambda$ with coefficients in $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, this must have solutions $\lambda \in \mathbb{C}$, and so $\lambda$ is an eigenvalue for $T$. (Thus an eigenvalue exists.)

**Theorem.** A field $k$ is algebraically closed if and only if every square matrix with coefficients in $k$ has an eigenvector.

(b) Give an example of a finite-dimensional vector space $V \neq 0$ over $\mathbb{R}$ and a linear transformation $T : V \to V$ which does not have an eigenvector.

Any matrix $T$ so that $\det(T-\lambda I_n)$ has no real roots $\lambda$ will work. In particular, $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has $\det(T-\lambda I_2) = \lambda^2 + 1$, and so $T$ has no eigenvalues in $\mathbb{R}$ and therefore no eigenvectors.

(c) Does a linear transformation of an infinite-dimensional vector space have to have an eigenvector? Either prove this is the case, or give an example of a linear transformation of an infinite-dimensional vector space which has no eigenvector.

If we don’t assume $V$ is over an algebraically closed field, then any example for part (b) would work: take a transformation on a finite subspace, which has no eigenvectors, and then define the transformation to fix $V$ outside of this subspace.

Note that our proof of part (a) does not work because $\det(T-\lambda I)$ may be an infinite power series, rather than a polynomial.

(d) Suppose that $T$ and $U$ are two linear transformations of a finite-dimensional vector space $V$ over $\mathbb{C}$ which commute with each other. Prove that there is some $v \in V$ which is an eigenvector for both $T$ and $U$.

Let $v$ be an eigenvector for $T$ with eigenvalue $\lambda$. Then $U(v)$ is also an eigenvector of $T$ with eigenvalue $\lambda$ since

$$T(U(v)) = U(T(v)) = U(\lambda v) = \lambda U(v).$$

In particular, the $\lambda$-eigenspace of $T_1$, call it $V_\lambda$, is invariant under $U$.

Thus the map $U_\lambda = U|_{V_\lambda}$ is a linear transformation $V_\lambda \to V_\lambda$, and so $U_\lambda$ has an eigenvector $w$.

Since $U_\lambda(w) = U(w)$, $w$ is an eigenvector of $U$, and since we $V_\lambda$, $w$ is also an eigenvector of $T$. 

Note that in an infinite-dimensional vector space, every vector is a finite linear combination of basis elements.
3. (August 1999 Problem 4) In this problem, all matrices are viewed over the complex numbers.

(a) For which complex numbers \( x \), if any, is the matrix
\[
\begin{pmatrix}
1 & -2 \\
8 & x
\end{pmatrix}
\]
not similar to a diagonal matrix? Explain.

The characteristic polynomial is \( \det(A-xI) = (-x)(x-9)+16 = x^2 - (1+9)x + (9+16) \).

The discriminant is \( (1+9)^2 - 4(9+16) = 100 - 80 = 20 \), which has roots \( x=9 \), \( x=-7 \).

Thus we can say when \( x=9, -7 \), the discriminant of the characteristic polynomial is nonzero, and thus \( A \) has distinct eigenvalues, which implies \( A \) is diagonalizable.

So we only need to consider the cases \( x=9 \) and \( x=-7 \).

If \( x=9 \), then the characteristic polynomial of \( A \) is \( x^2 - 10x + 25 = (x-5)^2 \). So the minimal polynomial of \( A \) is \((x-5)^2 \) or \((x-5)^2 \). We have \( A-5I = \begin{pmatrix} 4 & -2 \\ 8 & 4 \end{pmatrix} \).

So the minimal polynomial must be \((x-5)^2 \), which means \( A \) cannot be diagonalizable.

Similarly, if \( x=-7 \), the characteristic polynomial of \( A \) is \( x^2 + 6x + 9 = (x+3)^2 \).

We have \( A+3I = \begin{pmatrix} 4 & -2 \\ 8 & 4 \end{pmatrix} \) \( \neq 0 \), thus the minimal polynomial has repeated roots, and \( A \) cannot be diagonalizable.

(b) Let \( J \) be the \( n \times n \) matrix all of whose entries are equal to 1. Find a diagonal matrix similar to \( J \), or prove that one does not exist.

First note that since the columns of \( J \) are equal, they are linearly dependent, and so \( J \) cannot be invertible. This implies that \( J \) has \( 0 \) as an eigenvalue.

A square matrix is invertible if and only if \( 0 \) is not an eigenvalue.

Rank-Nullity tells us that \( \text{dim ker } J + \text{dim im } J = n \).

Since the image of \( J \) has dimension 1, and since the \( 0 \)-eigenspace is just the kernel of \( J \), we can conclude that the dimension of the \( 0 \)-eigenspace is \( n-1 \).

Thus \( J \) is diagonalizable if and only if it has a non-zero eigenvalue, since in that case, the sum of the dimensions of the eigenspaces will be \( n \). This is indeed the case, since \( n \) is also an eigenvalue of \( J \).

Thus \( J \) is diagonalizable if and only if it has a non-zero eigenvalue, since in that case, the sum of the dimensions of the eigenspaces will be \( n \). This is indeed the case, since \( n \) is also an eigenvalue of \( J \).

We conclude that the vector space \( \mathbb{C}^n \) is spanned by the eigenspace of \( J \), and so \( J \) is similar to the diagonal matrix
\[
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]

The matrix representation of the transformation \( J \) with a basis consisting of eigenvectors.
4. (January 2003 Problem 4) Let \( V \) be a vector space over the field \( K \) and let \((,) : V \times V \to K\) be a bilinear form on \( V \).

(a) If \( V \) is finite dimensional and if \( W \) is a proper subspace of \( V \), show that there exists a nonzero vector \( v \in V \) with \((w,v) = 0\) for all \( w \in W \).

The bilinear form induces a linear map \( V \to W^\ast \) given by \( v \mapsto (v,) \).

The problem is equivalent to showing that the kernel of this map is nontrivial, i.e., that the induced map is not injective.

Since \( \dim W^\ast = \dim W < \dim V \), there can be no injective map \( V \to W^\ast \).

Here we are using that \( V \) is finite-dimensional.

(b) Now let \( V \) have an infinite basis \( \mathcal{B} \) and let \((,)\) be the unique bilinear form such that, for all \( a, b \in \mathcal{B} \), we have \((a,b) = 0\) if \( a \neq b \) and \((a,b) = 1\) if \( a = b \). If \( W \) is the subspace of \( V \) spanned by all vectors of the form \( a - b \) with \( a, b \in \mathcal{B} \), show that \( W \) is a proper subspace of \( V \) and that there is no non-zero vector \( v \in V \) with \((w,v) = 0\) for all \( w \in W \).

First we show that \( W \) must be proper. If not, then for any \( a \in \mathcal{B} \), there are \( \lambda_i \in K \) such that \( a = \sum \lambda_i (b_i - c_i) \) for a finite index set \( M \).

Now define \( e = \sum (b_i + c_i) \). Then we have

\[
(a,e) = \left( \sum \lambda_i (b_i - c_i), \sum (b_i + c_i) \right) = \sum \lambda_i \left( (b_i, b_i) + (b_i, c_i) \right) = \sum \lambda_i = 0
\]

This implies that the basis element \( a \) does not appear in the collection \( \{b_i, c_i \mid i \in M\} \), since

\[
\sum (a,b_i) = \sum (a,c_i) = (a, \sum (b_i + c_i)) = (a,e) = 0
\]

If this is the case, then

\[
(a,a) = \left( a, \sum \lambda_i (b_i - c_i) \right) = \sum \lambda_i (a,b_i) - \sum \lambda_i (a,c_i) = 0
\]

which is a contradiction. Thus we have shown that \( W \) is a proper subspace of \( V \).

Now we show that there can be no non-zero vector \( v \in V \) such that \((w,v) = 0\) for every \( w \in W \). Let \( v \in V \) be nonzero. Then \( v = \sum \lambda_i a_i \) for a finite index set \( N \), where \( a_i \in \mathcal{B} \) and \( \lambda_i \in K \setminus \{0\} \).

Since \( N \) is finite, there exists \( b \in \mathcal{B} \) with \( b \neq a_i \) for all \( i \in M \). Let \( w = a_i - b \). Then \( w \in W \) and

\[
(w,v) = (a_i - b, \sum \lambda_i a_i) = \sum \lambda_i (a_i - b, a_i) = \lambda_i (a_i - b, a_i) = \lambda_i \neq 0.
\]
5. (August 2003 Problem 4) Let $A$ be a real $n \times n$ matrix. We say that $A$ is a **difference of two squares** if there exist real $n \times n$ matrices $B$ and $C$ with $BC = CB = 0$ and $A = B^2 - C^2$.

(a) If $A$ is a diagonal matrix, show that it is a difference of two squares.

For a real number $x$, we can define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

Then, for any $x$, we have $x^+ - x^- = x$, $x^+ x^- = 0$, and $x^+ x^- \geq 0$.

For a matrix $A = (a_{ij})$, we can define matrices $B = (b_{ij})$ and $C = (c_{ij})$ so that $b_{ij} = \sqrt{a_{ij}}$ and $c_{ij} = \sqrt{a_{ij}}$.

Now $B$ and $C$ are both diagonal, so we have $B^2 = (b_{ij}^2) = (a_{ij}^+)$, $C^2 = (c_{ij}^2) = (a_{ij}^-)$, and so $A = B^2 - C^2$.

Also, since $(b_{ii} c_{ii})^2 = a_{ii}^+ a_{ii}^- = 0$, we get $BC = CB = 0$.

(b) If $A$ is a symmetric matrix that is not necessarily diagonal, again show that it is a difference of two squares.

We know by the real spectral theorem that $A$ is diagonalizable, say $A = PDP^{-1}$ for a diagonal matrix $D$.

By part (a), there are $B$ and $C$ with $BC = CB = 0$ and $D = B^2 - C^2$.

Then $(PBP^{-1})(PCP^{-1}) = P(BCP^{-1}) = 0$, and similarly $(PCP^{-1})(PBP^{-1}) = 0$.

Also, $(PBP^{-1})^2 - (PCP^{-1})^2 = PB^2 P^{-1} - PC^2 P^{-1} = P(B^2 - C^2) P^{-1} = P(D) P^{-1} = A$, thus $A$ is also a difference of two squares.

(c) Suppose $A$ is a difference of two squares, with corresponding matrices $B$ and $C$ as above. If $B$ has a non-zero real eigenvalue, prove that $A$ has a positive real eigenvalue.

Let $\lambda$ be a non-zero eigenvalue of $B$ with eigenvector $v$.

Then $B^2 v = \lambda^2 v$. Also, $C^2 v = \frac{1}{\lambda} C(a v) = \frac{1}{\lambda} C(B v) = \frac{1}{\lambda} C(C B) v = 0$.

We conclude that $A v = (B^2 - C^2) v = \lambda^2 v - 0 = \lambda^2 v$, so $v$ is an eigenvector of $A$ with positive eigenvalue $\lambda^2$. 

**Real Spectral Theorem:**

A real symmetric matrix is diagonalizable.
1. (August 2013 Problem 4) Let $T_1, \ldots, T_k$ be a collection of linear transformations which act irreducibly on a finite-dimensional $\mathbb{C}$-vector space $V$ (i.e. such that there is no nontrivial proper subspace $W$ such that $T_i W \subseteq W$ for all $i$). Suppose $S : V \to V$ is a linear transformation which commutes with each of $T_1, \ldots, T_k$. Show that $S$ is a scalar operator.

Since $\mathbb{C}$ is algebraically closed, every matrix over $\mathbb{C}$ has an eigenvector.

Let $v$ be an eigenvector of $S$ with eigenvalue $\lambda \in \mathbb{C}$, and let $V_\lambda$ be the $\lambda$-eigenspace in $V$. Note that $V_\lambda \neq 0$ (it contains $v$).

Now since $ST_i = T_i S$, we know that $T_i (V_\lambda) \subseteq V_\lambda$ for each $i = 1, \ldots, k$.

**Proof:** Let $w \in V_\lambda$. We show that $T_i (w)$ is a $\lambda$-eigenvector of $S$ for each $i$.

We have $S (T_i (w)) = T_i (S w) = T_i (\lambda w) = \lambda T_i (w)$.

Thus we have shown that $V_\lambda$ is invariant under each $T_i$, which implies $V_\lambda = V$.

So $S$ acts as the scalar operator $S v = \lambda v$ for all $v \in V$. 

Matrices which commute with each other preserve each other's eigenspaces.
2. (August 2013 Problem 5) Let $V$ be a $k$-vector space and $V^\vee$ be its dual vector space. Consider the map $\psi: V^\vee \otimes_k V \rightarrow \text{Hom}_k(V, V) = \text{End}(V)$ given by $\sum_i \phi_i \otimes v_i \mapsto f$ where $f(v) = \sum_i \phi_i(v)v_i$.

(a) Characterize the image of this map.

We claim that the image of $\psi$ is the subspace $E = \{\psi \in \text{End}(V) \mid \dim \psi(V) < \infty\}$ of linear transformations with finite-dimensional images (i.e., $E = \{\psi \in \text{End}(V) \mid \dim \psi(V) < \infty\}$).

First we show that $\text{im} \, \psi \subseteq E$. If $\{v_i\}$ is a basis of $V$, then an element of $V^\vee \otimes V$ can be written as a finite sum $\sum y_j \otimes v_j$ for some $y_j \in V^\vee$. If $f = \psi(\sum y_j \otimes v_j)$, then we have $f(v) = \sum_j y_j(v)v_j \in \text{Span}\{v_j\}$, so $f \in E$.

Now we show $E \subseteq \text{im} \, \psi$. Let $f \in E$. By Rank-Nullity, we can find a basis $\{w_i\}$ of $V^\perp$ for which $f$ is a (possibly infinite) basis of $\ker f$, and $\{f(v_i)\}$ is a (finite) basis for $\text{im} \, f$. Define $y_j \in V^\vee$ to be the functional which is 1 at $v_j$, but 0 at all other basis elements. Then we have

$$\psi(\sum y_j \otimes f(v_j))(w_j) = \sum_j y_j(w_j)f(v_j) = f(w_j)$$

and

$$\psi(\sum y_j \otimes f(v_j))(v_j) = \sum_j y_j(v_j)f(v_j) = f(v_j).$$

We conclude that $\psi(\sum y_j \otimes f(v_j)) = f$, and so $E \subseteq \text{im} \, \psi$.

(b) Fill in the blank, and prove your answer: “The above map is an isomorphism if and only if $V$ is ______.”

$\psi$ is an isomorphism if and only if $V$ is finite-dimensional.

First, assume $V$ is finite-dimensional. Then every endomorphism of $V$ has finite-dimensional image, so $\psi$ must be surjective. Moreover, since $\dim (V^\vee \otimes V) = \dim (V^\vee) \cdot \dim (V) = \dim (V)^2 = \dim (\text{End}(V))$, we can conclude that $\psi: V^\vee \otimes V \rightarrow \text{End}(V)$ is in fact an isomorphism.

Conversely, suppose $\psi$ is an isomorphism. Then every endomorphism on $V$ has finite-dimensional image. In particular, the identity map on $V$ has finite-dimensional image, so $V$ must be finite-dimensional.
Note that in general, the tensor product $M \otimes N$ of $R$-modules $M, N$ (right, left respectively) is simply an abelian group with added structure, but it is not necessarily an $R$-module if $R$ is not abelian. However, it will be an $S$-module for any subring $S$ contained in the center of $R$. This gives the structure of $V^* \otimes_{End(V)} V$ as a $k$-vector space, where scalar multiplication has the form $\lambda \sum_{i} q_{i} \otimes v_{i} = \sum_{i} q_i (x_i) \otimes v_i = \sum q_i x_i \otimes v_i$.

Let $v_1, \ldots, v_n$ be a basis for $V$, and let $y_1, \ldots, y_n$ be the dual basis, so $y_i(v_j) = 1$, and $y_i(v_k) = 0$ for $i \neq j$. We claim that the set $\{y_i \otimes v_j\}$ spans $V^* \otimes_{End(V)} V$ as a $k$-vector space. For general tensor products, we know that every element can be written as a sum of pure tensors if $y_i \otimes v_j$ is a pure tensor, then $y_i \otimes v_j = \sum_{i} a_i \sigma_{ij}$, and so we can write each tensor as a linear combination of pure tensors of the form $y_i \otimes v_j$.

Fix $i, j$ with $i \neq j$. We show that $y_i \otimes v_j = 0$, and $y_i \otimes v_i = y_i \otimes v_i$.

Let $T \in End(V)$ be the transformation which sends $v_i \mapsto v_j$, and sends all other basis elements to 0. Define $S \in End(V)$ similarly, with $v_j \mapsto v_i$. Then we have

$$y_i \otimes v_j = y_i \otimes Tv_i = y_i \otimes v_j = 0 \otimes v_i = 0$$

$$y_i \otimes v_i = y_i \otimes Sv_j = y_i \otimes v_j = y_i \otimes v_i$$

Thus we have shown that the element $y_i \otimes v_i$ spans $V^* \otimes V$ as a $k$-vector space. This implies the dimension is at most 1. To finish the proof, we construct a surjective map of $k$-vector spaces $V^* \otimes V \rightarrow k$.

Define $\psi : V^* \otimes V \rightarrow k$ by $\psi(y_v) = \langle y_v, v \rangle$. This is clearly linear in both variables, and it is $End(V)$-bilinear since, if $T \in End(V)$, we have

$$\psi(Ty_v) = \langle y_v, Tv \rangle = \langle y_v, v \rangle = \psi(y_v).$$

Thus by the universal property of tensor products, $\psi$ induces a map $V^* \otimes_{End(V)} V \rightarrow k$.

Since $\psi(y_v) = \langle y_v, v \rangle = 1$, we see that $\psi(y_v) \neq 0$, and so $y_v \otimes v$ must have dimension 1 as a $k$-vector space, i.e. $V^* \otimes_{End(V)} V = k$.

$\psi$ is actually an isomorphism of vector spaces, which also implies the result, but we haven't shown that explicitly.
3. (January 1991 Problem 4) Let $F$ be an algebraically closed field of prime characteristic $p$ and let $V$ be an $F$-vector space of dimension precisely $p$. Suppose $A$ and $B$ are linear operators on $V$ such that $AB - BA = B$. If $B$ is nonsingular, prove that $V$ has a basis $\{v_1, \ldots, v_p\}$ of eigenvectors of $A$ such that $Bv_i = v_{i+1}$ for $1 \leq i \leq p-1$ and $Bv_p = \lambda v_1$ for some $\lambda \in F - \{0\}$.

Since $F$ is algebraically closed, $A$ has an eigenvector $v_1 \in V$, with eigenvalue $\lambda \in F$. We want to show that $Bv_1$ is also an eigenvector of $A$.

We have

$$ABv_1 = BAv_1 = B(Av_1) = B(\lambda v_1) + Bv_1 = (\lambda + 1)Bv_1.$$

So $Bv_1$ is an eigenvector of $A$, with eigenvalue $\lambda \neq \lambda + 1$.

Proceeding inductively, we see that $v_n = B^{n-1}v_1$ is an eigenvector with eigenvalue $\lambda + n - 1$. Also we clearly see that $Bv_n = v_{n+1}$ for each $n$. For $1 \leq n \leq p-1$, each eigenvalue is distinct, and so $v_1, \ldots, v_p$ are linearly independent, and therefore form a basis for $V$.

Thus each eigenspace has dimension 1, and so since $v_p$, $v_p$ both have eigenvalue $\lambda$, they must be scalar multiples of each other, say $v_p = \alpha v_1$. Since $v_p = Bv_p$, we are done.
5. (January 1994 Problem 4) Let $X$ be a subspace of $M_n(\mathbb{C})$, the $\mathbb{C}$-vector space of all $n \times n$ complex matrices. Assume that every nonzero matrix in $X$ is invertible. Prove that $\dim_{\mathbb{C}} X \leq 1$.

Let $A, B \in X$. Then $B^t A \in M_n(\mathbb{C})$. $B^t A$ has an eigenvector $v$ with eigenvalue $\lambda$.

So we have $B^t A v = \lambda v$, so $A v = \lambda B v$. This gives $(A - \lambda B) v = 0$, so $v$ is an eigenvalue of $A - \lambda B$ with eigenvalue $0$.

Thus $A - \lambda B$ cannot be invertible, since it has $0$ as an eigenvalue. But $A - \lambda B \in X$, and so by assumption we must have $A - \lambda B = 0$.

It follows that $A = \lambda B$, and so every matrix in $X$ is a scalar multiple of $B$, and therefore $X$ has dimension at most $1$. 
1. (January 1995 Problem 4) Let $A$ be an $n \times n$ matrix over an algebraically closed field $K$, and let $K[A]$ denote the $K$-linear span of the matrices $I = A^0, A^1, A^2, \ldots$. Show that $A$ is diagonalizable if and only if $K[A]$ has no non-zero nilpotents.

Since $K$ is algebraically closed, we can put $A$ in Jordan form, with $PAP^{-1} = J$. Note that any element of $K[A]$ has the form $f(A)$ for a polynomial $f(x) \in K[x]$. Note that $f(PAP^{-1}) = Pf(A)P^{-1}$, and so $f(PAP^{-1})$ is nilpotent if and only if $f(A)$ is nilpotent. Thus without loss of generality, we assume that $A$ is in Jordan form.

Let $m(x)$ be the minimal polynomial of $A$ Then $m(x) = \prod_j (x - \lambda_j)^{r_j}$, where $\lambda_j$ are distinct eigenvalues of $A$, and $r_j$ is the size of the largest Jordan block associated to $\lambda_j$.

For each $\lambda_j$, the geometric and algebraic multiplicities are equal if and only if each Jordan block associated to $\lambda_j$ have size 1, i.e. if and only if $r_j = 1$.

Thus we can conclude that $A$ is diagonalizable if and only if $r_j = 1$ for each $\lambda_j$, i.e. if and only if the minimal polynomial has distinct factors.

Define $f(x) = \prod (x - \lambda_j)$, i.e. $f(x)$ has each $\lambda_j$ as a root with multiplicity 1. Certainly $f(x)|m(x)$.

If $r = \max_i \{r_i\}$, then $m(x) | f(x)^r$.

If $r = 1$, then $f(A) = 0$, but $f(A)^r = 0$, so $f(A)$ is nilpotent.

Conversely, any nilpotent $g(A) \in K[A]$ with $g(A)^r = 0$ must have $m(x) | g(x)^r$, so $f(x) | g(x)$.

If $r = 1$, then $f(x) = m(x)$, and so $g(A) = 0$. 

A is diagonalizable if and only if, for every eigenvalue $\lambda$ of $A$, its geometric and algebraic multiplicities coincide.