Show all work. Circle your answer.

No notes, no books, no calculator, no cell phones, no pagers, no electronic devices at all.

Solutions will be posted shortly after the exam: www.math.wisc.edu/∼miller/m240

Name______________________________________

Problem	Points	Score
1	6	
2	6	
3	6	
4	6	
5	6	
6	6	
7	6	
8	5	
9	5	
10	8	
11	8	
12	8	
13	8	
14	8	
15	8	
Total	100	

1. (6 pts) Show that $p \rightarrow q$ and $(\neg q) \rightarrow (\neg p)$ are logically equivalent.

2. (6 pts) Construct a truth table for the following propositional sentence:

$$(p \vee q) \rightarrow (p \wedge r)$$

3. (6 pts) Find a statement which is logically equivalent to

$$\neg[\ (\exists x\ P(x)) \to (\forall y\ Q(y))\ ]$$

but in which the negation sign appears (if at all) only in front of the predicate symbols.

4. (6 pts) Let $A = \{1, 2\}$ and $B = \{1, 3, 5\}$. Find $A \times B$.

5. (6 pts) Let $A = \{1, 3, 4\}$, $B = \{2, 4\}$, and $C = \{3, 4, 5, 6\}$. Find

(a) $|A|$

(b) $B \setminus C$

(c) $(A \cup B) \cap C$

6. (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \lfloor x \rfloor$$

Let $S = \{2, 5\}$. Find $f^{-1}(S)$.

7. (6 pts) Compute the following double sum:

$$\sum_{i=1}^{2} \sum_{j=1}^{3} (i+j)$$

8. (5 pts) How large a problem can be solved in 10 seconds or less using an algorithm that when input n requires $f(n) = n^2$ bit operations where each bit operation is carried out in 10^{-9} seconds?

9. (5 pts) The value of the Euler ϕ-function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. Find $\phi(12)$.

10. (8 pts) What is smallest positive integer k such that the function

$$f(n) = (n\log(n) + 1)^2$$

is $\mathcal{O}(n^k)$?

11. (8 pts) Show that $2^n > 2n + 1$ for all integers $n \geq 3$.

12. (8 pts) Let f_n be the n^{th} element of the Fibonacci sequence. Prove that

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$$

for every positive integer n.

13. (8 pts) Find a 2×2 matrix A so that

$$A \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

Hint: Solve a system of linear equations.

14. (8 pts) Let $n = (\ldots a_k a_{k-1} \ldots a_2 a_1 a_0)_2$ be any positive integer written in binary.
Let E be the set of even k such that $a_k = 1$ and
let O be the set of odd k such that $a_k = 1$.
Show that n is divisible by 3 if and only if $|E| + 2|O|$ is divisible by 3.

15. (8 pts)

(a) Find $d = \gcd(54, 17)$ using the Euclidean Algorithm.

(b) Find integers a and b such that $d = a\,54 \;+\; b\,17$

Answers

1. $p \rightarrow q$ is false iff p is true and q is false.
$(\neg q) \rightarrow (\neg p)$ is false iff $\neg q$ is true and $\neg p$ is false.
Hence they have the same truth table.

2.

p	q	r	$(p \vee q)$	$\rightarrow$	$(p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	T	F

3. $(\exists x\ P(x)) \wedge (\exists y\ \neg Q(x))$

4. $\{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5)\}$

5. $|A| = 3$, $B \setminus C = \{2\}$, $(A \cup B) \cap C = \{3,4\}$

6. $\{x : 2 \leq x < 3 \text{ or } 5 \leq x < 6\} = [2,3) \cup [5,6)$

7. 21

8. $n = 10^5$

9. 4

10. $k = 3$ because $\log(n)$ is $\mathcal{O}(n^\epsilon)$ for any real $\epsilon > 0$.

11. This is proved by induction.
Basis: $n = 3$ this true because $2^3 = 8 > 7 = 2 * 3 + 1 = 7$
Inductive step. Assume $2^n > 2n + 1$ and $n \geq 3$. Then

$$2^{n+1} = 2 * 2^n = 2^n + 2^n > 2n + 1 + 2^n > 2n + 1 + 8 > 2(n+1) + 1$$

By inductive hypothesis and since $2^n \geq 2^3 = 8$. Hence

$$2^{n+1} > 2(n+1) + 1$$

as we needed to show.

12. This is proved by induction.
Basis: $f_1 = 1 = f_2$
Inductive Step: Assume true for n. Then

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} + f_{2n+1} = f_{2n} + f_{2n+1}$$

by inductive hypothesis. But by the definition of Fibonacci sequence

$$f_{2n} + f_{2n+1} = f_{2n+2} = f_{2(n+1)}$$

and so noting that $2(n+1) - 1 = 2n + 1$

$$f_1 + f_3 + f_5 + \cdots + f_{2(n+1)-1} = f_{2(n+1)}$$

as was to be shown.

13.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

14. If k is even, say $k = 2l$ then

$$2^k = 2^{2l} = 4^l$$

But $4 \equiv_3 1$ so it follows that

$$2^k \equiv_3 4^l \equiv_3 1^l \equiv_3 1$$

If k is odd, then $k - 1$ is even and hence

$$2^k = 2 * 2^{k-1} \equiv_3 2$$

Now since $n = \sum_{k \in E} 2^k + \sum_{k \in O} 2^k$ it follows from above that

$$n \equiv_3 \sum_{k \in E} 1 + \sum_{k \in O} 2$$

but

$$\sum_{k \in E} 1 + \sum_{k \in O} 2 = |E| + 2|O|$$

so 3 divides n iff $n \equiv_3 0$ iff $|E| + 2|O| \equiv_3 0$ iff 3 divides $|E| + 2|O|$

15.
$54, 17 \quad 54 = 3(17) + 3$
$3, 17 \quad 17 = 5(3) + 2$
$3, 2$ and we see that gcd is 1.
$1 = -1(3) + 2(2)$ using $2 = 17 - 5(3)$ we get
$1 = -1(3) + 2(17 - 5(3)) = -11(3) + 2(17)$
$1 = -11(3) + 2(17)$ using 3=54-3(17) we get
$1 = -11(54 - 3(17)) + 2(17) = -11(54) + 35(17)$