Show all work. Circle your answer.

No notes, no books, no calculator, no cell phones, no pagers, no electronic devices at all.

Solutions will be posted shortly after the exam: www.math.wisc.edu/∼miller/m240

Name____________________________________

Problem	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	4	
10	8	
11	8	
12	8	
13	8	
Total	100	

1. (8 pts) How many strings of 8 ASCII characters contain the character % at least once? Note that there are 128 ASCII characters.

2. (8 pts) Let x be a positive real number and n any positive integer. Show that there is a positive integer j such that the distance from jx and some integer is $\leq \frac{1}{n}$.

Hint: For each j there are integers k and m with $0 \leq m \leq n-1$ so that

$$k + \frac{m}{n} \leq jx \leq k + \frac{m+1}{n}$$

3. (8 pts) One hundred tickets numbered $1, 2, \ldots, 100$ are to be sold to 100 different people. Eleven prizes are awarded, a grand prize (a scuba diving trip to the Caribbean island of Bonaire), and 10 consolation prizes (a round-trip bus ticket to Milwaukee not including taxes or gratuities).

How many ways are there to distribute the prizes? Assume each person can win at most one prize.

4. (8 pts) Ten sections from a book are to be tested on the second midterm. Each section of the book ends with 20 exercises. The instructor randomly chooses one exercise from each section to put on the exam.

(a) What is the probability that every problem chosen is exercise 17?

(b) What is the probability that every problem chosen is exercise 17 except exactly one?

5. (8 pts) How many strings of decimal digits (0-9) are there of length twenty which contain exactly one zero, two ones, and three twos?

6. (8 pts) Find a recurrence relation for the number, S_n, of decimal (0-9) strings of length n which contain at least two consecutive digits which are both 9's.

7. (8 pts) Determine values of the constants A and B so that $a_n = An + B$ is a particular solution of $a_n = a_{n-1} - 2a_{n-2} + n$.

8. (8 pts) How many elements are in the union of four sets if the sets have 100,200,300,400 elements respectively, each pair of sets has 50 elements in common, each triple has 10 elements in common, and there is exactly one element in all four sets?

9. (4 pts) Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f. That is $R = \{(a, f(a)) \ : \ a \in A\}$. What is R^{-1}?

10. (8 pts) Let

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

represent the relation R. Find a matrix which represents $R^2 = R \circ R$.

11. (8 pts) Let $R \subseteq \{1,2,3,4\} \times \{1,2,3,4\}$ be the following binary relation:

$$R = \{(1,3),(4,4),(4,2),(3,4),(2,1)\}$$

Draw a digraph picture for R. Find a path of length 6 starting at 1 and ending at 2.

12. (8 pts) For any $n = 2, 3, 4, \ldots, 15$ define $f(n) = p$ where p is the least prime which divides n. Define the relation $x \equiv y$ on this set by $x \equiv y$ iff $f(x) = f(y)$.

(a) Prove that $\equiv$ is an equivalence relation on the set $\{2, 3, 4, \ldots, 15\}$.

(b) What are its equivalence classes?

13. (8 pts) Draw the Hasse Diagram for the partial order $(P(\{a, b, c\}, \subseteq))$.

Answers

1. (4.1 - 17) There are 128^8 strings all together. 127^8 strings have no occurence of % in them. Hence $128^8 - 127^8$ is the number in which % occurs at least once.

2. (4.2 - 17) The hint is true because for any real number jx there is an integer k with $k \leq jx \leq k+1$. We then divide the interval $[k, k+1]$ into n subintervals of length $\frac{1}{n}$ to see that we must have some m.

By the pidgeon hole principle we can find $j < j'$ that have the same m as in the hint, i.e.

$$k + \frac{m}{n} \leq jx \leq k + \frac{m+1}{n}$$
$$k' + \frac{m}{n} \leq j'x \leq k' + \frac{m+1}{n}$$

for some integers k, k'. This is because there are infinitely many j's but only n possible $m's$. We claim that $(j'-j)x$ is within $\frac{1}{n}$ of the integer $k'-k$.

Since negation reverses inequalities

$$-k - \frac{m+1}{n} \leq -jx \leq -k - \frac{m}{n}$$

So that adding the last two inequalities we get:

$$k' - k - \frac{1}{n} \leq (j'-j)x \leq k' - k + \frac{1}{n}$$

So $(j'-j)x$ is within $\frac{1}{n}$ of the integer $k'-k$.

3. (4.3-17) There are 100 choices for the grand prize. To award the bus tickets we must then choose 10 tickets out of the remaining 99. Hence the answer is

$$100 \times \binom{99}{10}$$

4. (4.4- ??)

(a) There are 20^{10} possible exams. Only one has problem 17 of each section on it. So

$$\frac{1}{20^{10}}$$

(b) There are 10×19 problems which are not 17. Hence there are 190 exams as described. So

$$\frac{190}{20^{10}}$$

5. (4.6-17)

Choose one digit to be the zero, $\binom{20}{1}$ two of the remaining to be ones $\binom{19}{2}$ and three of the remaing to be twos $\binom{17}{3}$ the other 14 digits can be taken arbitrarily from the 7 digits 3-9. There are 7^{14} ways to do this.

$$\binom{20}{1}\binom{19}{2}\binom{17}{3}7^{14}$$

6. (5.1-17) The strings of length $n+1 \geq 3$ which contain at least two consecutive 9's can divided as follows. Either the
first two digits are 99, there are 10^{n-1} such strings
or the first digit is not a 9, there are $9S_n$ such strings
or the first digit is a 9 and the second is not, there are $9S_{n-1}$ of these.

Hence

$$S_{n+1} = 10^{n-1} + 9S_n + 9S_{n-1}$$

7. (5.2-25)

$$\begin{aligned}
An + B &= A(n-1) + B - 2(A(n-2) + B) + n \\
An + B &= An - A + B - 2An + 4A - 2B + n \\
An + B &= (A - 2A + 1)n - A + B + 4A - 2B \\
An + B &= (-A + 1)n - A + B + 4A - 2B \\
0 &= (-2A + 1)n + (3A - 2B)
\end{aligned}$$

For this to be true for every n we must have that $-2A+1=0$ and $3A-2B=0$. So $A = \frac{1}{2}$ and so $B = \frac{3}{4}$. Hence $a_n = \frac{1}{2}n + \frac{3}{4}$

8. (5.5-17)

$$100 + 200 + 300 + 400 - \binom{4}{2}50 + \binom{4}{3}10 - 1 = 739$$

9. (6.1-17)

It is the graph of the inverse function or

$$R^{-1} = \{(f(a), a) \ : \ a \in A\}$$

10. (6.3-9)

$$M_{R^2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

11. (6.4-16,17)

1,3,4,4,4,4,2 is a path of length six.

12. (6.5-6,17)

(a) A relation is an equivalence relation iff it is reflexive, symmetric and transitive.
$x \equiv x$ iff $f(x) = f(x)$ so it is reflexive
$x \equiv y$ implies $f(x) = f(y)$ implies $f(y) = f(x)$ implies $y \equiv x$ so it is symmetric
$x \equiv y$ and $y \equiv z$ imply $f(x) = f(y) = f(z)$ so $x \equiv z$. so this relation is transitive.

(b) 2 is the smallest prime. The equivalence classes are
$\{2, 4, 6, 8, 10, 12, 14\}$
$\{3, 9, 15\}$
$\{5\}$
$\{7\}$
$\{11\}$
$\{13\}$

13. (6.6)

See text page 421.