Show all work.

No notes, no books, no calculators, no cell phones, no pagers, no electronic devices of any kind.

Name____

Circle your Discussion Section:

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DIS 303 12:05p T B235 VAN VLECK
DIS 304 12:05p R B235 VAN VLECK
DIS 307  2:25p T B139 VAN VLECK
DIS 308  2:25p R B309 VAN VLECK
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Problem	Points	Score
1	4	
2	4	
3	6	
4	6	
5	8	
6	7	
7	7	
8	6	
9	7	
10	8	
11	8	
12	7	
13	7	
14	8	
15	7	
Total	100	

Solutions will be posted shortly after the exam: www.math.wisc.edu/~miller/m240

1. (4 pts) Construct a truth table for the compound proposition:

$$(p \to q) \vee (\neg p \to q)$$

2. (4 pts) Use a truth table to verify:

$$(p \to q) \equiv (\neg q \to \neg p)$$

3. (6 pts) Let $P(x)$ be the statement $x + 1 > x^2$ and suppose that the universe of discourse consists of the integers. What are the truth values of the following?

1. $P(0)$
2. $P(1)$
3. $P(-1)$
4. $\exists x\ P(x)$
5. $\forall x\ P(x)$
6. $\forall x \exists y\ \ ((y > x) \wedge P(y))$

4. (6 pts) Determine the truth value of each of the following if the universe of discourse for all variables consists of the positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

1. $\forall n\ \exists m \quad n^2 < m$
2. $\exists m\ \forall n \quad n^2 < m$
3. $\exists n\ \exists m \quad n^2 + m^2 = 5^2$
4. $\exists n\ \exists m \quad n^2 + m^2 = 6^2$
5. $\forall n\ \forall m \quad (n \leq m\ \vee\ m \leq n)$
6. $\forall n\ \forall m \quad (n < m\ \vee\ m < n)$

5. (8 pts) Determine if the following arguments are correct. If it is correct, what rule of inference is being used. If it is not, what logical error occurs?

(a) If n is an integer with $n \geq 2$, then $n^3 \geq 8$. Suppose $n < 2$. Then $n^3 < 8$.

(b) If n is an integer with $n > 2$, then $n^3 > 8$. Suppose $n^3 \leq 8$. Then $n \leq 2$.

6. (7 pts) How many different elements does $A \times A \times A$ have if A has n elements?

7. (7 pts) What can we say about the sets A and B if $A \oplus B = \emptyset$. The symbol $\oplus$ denotes the symmetric difference.

8. (6 pts) Let $h(x) = \lceil x \rceil$. Find

1. $h^{-1}(\{2\})$

2. $h^{-1}(\{x \ : \ -1 \leq x \leq 1\})$

3. $h(\{x \ : \ -1 \leq x \leq 1\})$

9. (7 pts) Use the bubble sort to sort the list $3, 2, 4, 5, 1$ showing the lists obtained at each step, i.e., after each time you do a comparison.

10. (8 pts) Find the least integer n such that $f(x)$ is $O(x^n)$ where

$$f(x) = \frac{2x^5 + x^2 + 1}{3x^2 + 4x\ln(x)}$$

11. (8 pts) Show that if $2^n - 1$ is prime, then n is prime.

    Hint: $(x^m - 1) = (x - 1)(x^{m-1} + x^{m-2} + \cdots + x + 1)$

12. (7 pts) Convert the integer 11001111 from binary notation to decimal notation.

13. (7 pts) How much time does an algorithm using 2^{40} bit operations take if each bit operation takes 10^{-9} seconds?

14. (8 pts) Suppose that an integer a is not divisible by the prime p. Show that no two of the integers:

$$a, 2a, 3a, \ldots, (p-1)a$$

are congruent modulo p.

15. (7 pts) Find AB if

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 2 & 0 \end{bmatrix}$$

Answers

1. 1.1-27
This is a tautology.

2. 1.2-3
This is the contrapositive.

3. 1.3-11
TTFTFF

4. 1.4-27
TFTFTF

5. 1.5-13
(a) The logical form of this argument is:

$$\begin{array}{l} P \to Q \\ \neg P \\ \hline \neg Q. \end{array}$$

This is an incorrect inference even though it reaches a correct conclusion.

(b) The logical form of this argument is:

$$\begin{array}{l} P \to Q \\ \neg Q \\ \hline \neg P. \end{array}$$

This is a correct logical inference.

6. 1.6-25
n^3.

7. 1.7-31
$A = B$

8. 1.8-35
1. $(1, 2]$
2. $(-2, 1]$
3. $\{-1, 0, 1\}$

9. 2.1-35
$\underline{32}451$
$2\underline{34}51$
$23\underline{45}1$
$234\underline{51}$
$\underline{23}415$
$2\underline{34}15$

$23\underline{41}5$
$231\underline{45}$
$\underline{23}145$
$2\underline{31}45$
$21\underline{34}5$
$213\underline{45}$
$\underline{21}345$
12345
followed by the end of this pass and one more pass to check that nothing changes.

10. 2.2-7
$O(x^3)$

11. 2.4-23
Suppose that n is not prime and let $n = km$ for integers k, m with $1 < k, m < n$. Put $x = 2^k$ and using the hint note that

$$2^n - 1 = (2^k)^m - 1 = (x^m - 1) = (x-1)(x^{m-1} + x^{m-2} + \cdots + x + 1)$$

and so $2^n - 1$ is not prime.

12. 2.5-3
207

13. 2.3-11
$2^{40}10^{-9}$ seconds. A good estimate is to use $2^{10} = 1024 \approx 1000$ so

$$2^{40}10^{-9} = \frac{2^{40}}{10^9} = \frac{(2^{10})^4}{10^9} \approx \frac{(1000)^4}{10^9} = \frac{(10^3)^4}{10^9} = \frac{10^{12}}{10^9} = 10^3$$

14. 2.6-17
Suppose for contradiction that there are i, j integers with $1 \leq i < j \leq p-1$ such that

$$ia \equiv_p ja$$

Then

$$0 \equiv_p (j-i)a$$

and so p divides $(j-i)a$. Since p is prime and does not divide a it must divide $j-i$. But this is impossible because $1 \leq j-i < p$.

15. 2.7-3

$$\begin{bmatrix} -2 & -3 \\ -3 & 5 \end{bmatrix}$$

The following program was used to pick the problems on this test. In some cases the problem is identical and in others it is just similar.

```
#! /usr/ucb/python

import string
import sys
import random

f=open("hmwk1",'r')    # input file
lines=f.readlines()

random.seed("the three stooges")

for line in lines:
  s=string.split(line)
  if len(s)> 4:
    section=s.pop(0)
    print random.choice(s).rjust(2) + "  "+string.lstrip(line)
```