Show all work.

No notes, no books, no calculators, no cell phones, no pagers, no electronic devices of any kind.

Name______________________________________

Circle your Discussion Section:

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DIS 303 12:05p T B235 VAN VLECK
DIS 304 12:05p R B235 VAN VLECK
DIS 307  2:25p T B139 VAN VLECK
DIS 308  2:25p R B309 VAN VLECK
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Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Solutions will be posted shortly after the exam: www.math.wisc.edu/∼miller/m240

1. (10 pts) Let p be a prime. Prove that for any integers a and b that

$$a^2 \equiv_p b^2 \quad \text{if and only if} \quad a \equiv_p b \text{ or } a \equiv_p -b.$$

$a \equiv_p b$ stands for $a = b \bmod p$, or equivalently, $a - b$ is divisible by p.

2. (10 pts) $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of positive integers. A nonempty set X is countable iff there exists a function $f : \mathbb{N} \to X$ which is onto. Equivalently, a nonempty set X is countable iff it can be enumerated:

$$X = \{x_1, x_2, x_3, \ldots\}$$

(a) Prove that the set E of even integers is countable. (Note: we include in E the negative even integers.)

(b) Prove exactly one of the following. If you choose to prove (2) you may assume (1) without proof.

1. The set of ordered pairs of natural numbers, $\mathbb{N} \times \mathbb{N}$, is countable.
2. The union of a countable family of countable sets is countable, i.e., if $\{X_n : n \in \mathbb{N}\}$ is a family of sets where each X_n is countable, then X is countable:

$$X = \bigcup \{X_n : n \in \mathbb{N}\}$$

3. (10 pts) Consider a sequence defined by $a_0 = 0, a_1 = 1$ and $a_n = 2a_{n-1} + 3a_{n-2} + n^2$ for $n \geq 2$. Write pseudocode for an iterative algorithm to compute $f(n) = a_n$. Iterative means that the algorithm may use loops (for-next, repeat-until, do-while, etc) but may not call itself recursively. Do not use an array to store the values of a_i for $i < n$.

4. (10 pts) What is the minimum number of people each of whom comes from one of 25 cities which guarantees there are at least 9 from the same city?

5. (10 pts) How many possible ways are there for first, second, third, and last place finishes in a car race with 20 cars? Assume no ties.

6. (10 pts) What is the coefficient of x^3 in the expansion of $(x-1)^8$?

7. (10 pts) How many ways can 10 cans of soup be put onto 4 distinguishable shelves

(a) If the cans are indistinguishable? For example, each can contains the same brand of chicken noodle soup.

(b) If each of the ten cans is a different kind of soup and the position of the cans on the shelves matter?

8. (10 pts) Suppose that there are 10 sections in a discrete math book and 10 assigned exercises from each section. An instructor randomly chooses 10 of the 100 assigned exercises to base his test on.

(a) What is the probability that all 10 chosen exercises come from the same section in the book?

(b) What is the probability that each chosen exercise is from a different section in the book?

9. (10 pts) Find the general solution to

$$a_{n+1} - 5a_n + 6a_{n-1} = 2n + 5$$

(Hint: There should be two arbitrary constants in your solution.)

10. (10 pts)

(a) Suppose that $f(n) = 7f(n/2)+n^2$ whenever n is divisible by 2. Find the general solution for the recurrence relation satisfied by $a_k = f(2^k)$.

(b) Suppose that f of part (a) is an increasing function positive function. Estimate the size of $f(n)$ using the big-O notation.

The following program was used to pick the problems on the test. I combined a couple and changed the probability problem completely.

```
#! /usr/ucb/python

import string
import sys
import random

f=open("hmwk2",'r')    # input file
lines=f.readlines()

random.seed("Moe, Larry, and Curly")

linenumb=0
problems=[]
for line in lines:
  if linenumb > 30:
    s=string.split(line)
    if len(s)>0:
      section=s.pop(0)
      for prob in s:
        problems.append(section+"-"+prob.rjust(2))
  linenumb=linenumb+1
random.shuffle(problems)
problems=problems[0:12]
problems.sort()
for prob in problems:
  print prob
```

Answers

1. The main point is that for a prime p if p divides $(a-b)(a+b)$, then either p divides $a-b$ or p divides $a+b$.

2.
(a) $\{0, 2, -2, 4, -4, \ldots\}$
(b) (1) See figure 2 on page 235.
(b) (2) Let $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be an onto function. Suppose $f_n : \mathbb{N} \to X_n$ is onto for each $n \in \mathbb{N}$. Define the function $h : \mathbb{N} \to X$ by $h(n) = x$ where $x = f_i(j)$ and $g(n) = (i, j)$. It is not hard to see that h is onto.

3. The inner code of the loop might look like this:

$savey = y$
$y = 2y + 3x + z^2$
$x = savey$
$z = z + 1$

Each iteration of the loop has the effect of 'shifting'

$y = a_n$, $x = a_{n-1}$, $z = n$
to
$y = a_{n+1}$, $x = a_n$, $z = n + 1$.

4. 201

5. $20 \cdot 19 \cdot 18 \cdot 17$.

6. -56

7. (a) $\binom{13}{3}$ (b) $10!\binom{13}{3}$

8.

$$(a) \quad \frac{10}{\binom{100}{10}}$$

$$(b) \quad \frac{10^{10}}{\binom{100}{10}}$$

9. $a_n = C2^n + D3^n + n + 5$

10.
(a) $a_k = C(7^k) + (-\frac{1}{3})4^k$ for an arbitrary constant C.
(b) $f(n)$ is $O(n^c)$ where $c = \log_2(7) \approx 2.8$.