1. Prove that well-ordering is not a pseudo-elementary class, i.e. show that there does not exist a first order theory $T$ in a language $L$ which includes the symbol $\leq$ (but may be much bigger), such that the class of well-orderings coincides with the reducts of models of $T$ to $\leq$.

2. Show that $L$ structures $A$ and $B$ are elementarily equivalent iff they have isomorphic elementary extensions.

3. A structure $A$ is finitely generated iff there exists a finite $F \subset A$ such that there is no proper substructure of $A$ containing $F$. Let $T$ be a first order theory in a countable language. Show that if $T$ has an infinite model then some countable model of $T$ is not finitely generated.

4. The language $L_{\kappa,\lambda}$ is defined as follows. The atomic formulas are the same as for first order logic. In addition to the usual formation rules for first order logic we have the following:

1. if $\Phi$ is a set of fewer than $\kappa$ formulas then $\bigwedge \Phi$ and $\bigvee \Phi$ are formulas; and

2. if $\theta$ is a formula and $\exists x_0 \exists x_1 \ldots \exists x_\delta$ and $\forall x_0 \forall x_1 \ldots \forall x_\delta$ are $\delta$ sequences of quantifiers where $\delta < \lambda$, then $\exists x_0 \exists x_1 \ldots \exists x_\delta \theta$ and $\forall x_0 \forall x_1 \ldots \forall x_\delta \theta$ are formulas.

Thus $L_{\omega,\omega}$ is ordinary first order logic. $L_{\infty,\infty}$ is the union of $L_{\kappa,\lambda}$ for all $\kappa$ and $\lambda$ and $L_{\infty,\omega}$ is the union of $L_{\kappa,\omega}$ for all $\kappa$.

Suppose $F \subset L_{\infty,\infty}$ is a set of formulas closed under subformula. Suppose that $A$ is an infinite $L$ structure, $X \subset A$, and $\kappa$ is cardinal such that

$$|X| + |F| + \aleph_0 \leq \kappa \leq |A|$$

$$\forall \theta(\bar{x}) \in F \quad \kappa^{|\bar{x}|} = \kappa$$

Show that there exists $B \preceq_F A$, $X \subset B$, and $|B| = \kappa$. $B \preceq_F A$ means elementary substructure with respect to all the formulas in $F$, i.e. for every formula $\theta \in F$ and sequence $\bar{b}$ from $B$ (may be of infinite length)

$$B \models \theta(\bar{b}) \iff A \models \theta(\bar{b})$$
5. The \( L \) structures \( \mathcal{A} \) and \( \mathcal{B} \) are \( L_{\infty,\omega} \) elementarily equivalent iff they satisfy the same sentences of \( L_{\infty,\omega} \). This is written \( \mathcal{A} \equiv_{\infty,\omega} \mathcal{B} \). Show that \( \mathcal{A} \equiv_{\infty,\omega} \mathcal{B} \) iff there exists a back and forth property between \( \mathcal{A} \) and \( \mathcal{B} \).

6. Let \( T_0 \) and \( T_1 \) be \( L \) theories such that for every finite \( F_0 \subset T_0 \) and \( F_1 \subset T_1 \) there are \( L \) structures \( \mathcal{A}_0 \models F_0 \) and \( \mathcal{A}_1 \models F_1 \) such that \( \mathcal{A}_0 \) is a substructure of \( \mathcal{A}_1 \). Show that there exists \( \mathcal{A}_0 \models T_0 \) and \( \mathcal{A}_1 \models T_1 \) such that \( \mathcal{A}_0 \) is a substructure of \( \mathcal{A}_1 \).

7. Prove that a first order theory \( T \) is axiomatizable by \( \exists \) sentences iff every superstructure of a model of \( T \) is a model of \( T \).

8. Show that \( \exists \forall \)-sentences are preserved by sandwiches, i.e. for any \( \theta \) a \( \exists \forall \)-sentence and models \( \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \) if \( \mathcal{A} \preceq \mathcal{C} \models \theta \) then \( \mathcal{B} \models \theta \).

9. Show that there exists no interpolant \( \rho \) in \( BA(\forall) \) for \( \vdash \exists x \forall y \ R(x, y) \rightarrow \forall y \exists x \ R(x, y) \)

10. Show that a theory \( T \) is \( \forall \exists \forall \)-axiomatizable iff for every sequence

\[
\mathcal{A}_0 \preceq \forall \mathcal{A}_1 \preceq \forall \mathcal{A}_2 \preceq \ldots
\]

if each \( \mathcal{A}_n \models T \), then \( \bigcup_{n \in \omega} \mathcal{A}_n \models T \).

11. Suppose \( \vdash \theta \rightarrow \psi \) where \( \theta \) is a \( \forall \exists \forall \)-sentence and \( \psi \) is a \( \exists \forall \)-sentence. Show that there exists an interpolant \( \rho \) which is in \( BA(\forall) \).

12. This is a counterexample to a Beth definability theorem for structures. Consider the stucture \((\omega, S, 0)\) where \( S \) is the successor function and 0 is the constant zero. Let \( R \) be a binary operation symbol. Show that < is implicitly definable in the structure \((\omega, S, 0)\), i.e. there exists a sentence \( \theta \) in the language of \( S, 0, R \) such that for every \( R \subseteq \omega^2 \)

\[
(\omega, S, 0, R) \models \theta \text{ iff } R = \{(x, y) \in \omega^2 \mid x < y\}
\]

But show that it is not explicitely definable, i.e. there does not exist \( \psi(x, y) \) a formula in the language of \( S, 0 \) such that for every \( x, y \in \omega \)

\[
[(\omega, S, 0) \models \psi(x, y)] \text{ iff } x < y
\]

13. Suppose \( T_0 \) is an \( L_0 \) theory and \( T_1 \) is an \( L_1 \) theory such that \( T_0 \cup T_1 \) is inconsistent. Prove the there exists a \( L_0 \cap L_1 \)-sentence \( \theta \) such that \( T_0 \vdash \theta \) and \( T_1 \vdash \neg \theta \).
14. Show that if $T$ is a first order theory with arbitrarily large finite models, then $T$ has a model of cardinality the continuum.

15. Give an example of a first order theory $T$ with arbitrarily large finite models, but every infinite model of $T$ has cardinality at least the continuum.

16. Suppose $f : I \mapsto J$, $U$ is an ultrafilter on $I$, and

$$V = \{ X \subseteq J : f^{-1}(X) \in U \}$$

Show that $V$ is an ultrafilter on $J$. This is the definition of the Rudin-Keisler ordering on ultrafilters and is written $V \leq_{\text{RK}} U$. Show that for any structure $\mathcal{A}$ there is an elementary embedding of $\mathcal{A}/V$ into $\mathcal{A}/U$.

17. An ultrafilter $U$ on $I$ is called $(\kappa, \omega)$-regular iff there exists $I_\alpha \in U$ for $\alpha < \kappa$ such that for any infinite $X \subseteq \kappa$

$$\bigcap\{ I_\alpha : \alpha \in X \} = \emptyset$$

Show that $U$ is $(\kappa, \omega)$-regular iff there exists a regular ultrafilter $V$ on $[\kappa]^{<\omega}$ such that $V \leq_{\text{RK}} U$. $V$ regular means that for every $\alpha < \kappa$

$$\{ F \in [\kappa]^{<\omega} : \alpha \in F \} \in V$$

18. Let $U$ be a $(\kappa, \omega)$-regular ultrafilter on $I$ ($\kappa$ an infinite cardinal) and let $\mathcal{A}$ be any infinite $L$-structure where $|L| = \kappa$. Show that $\mathcal{A}/U$ is weakly saturated (i.e. realizes every consistent type).

19. Show that if $\mathcal{A} = (\omega_1, <)$ then $\mathcal{A}/\omega$ is not $\omega_\omega$-saturated for any ultrafilter $U$ on $\omega$.

20. Let $T$ be the theory of $(P(X), \subseteq)$ where $X$ is an infinite set, $P(X)$ is the power set of $X$, and $\subseteq$ is the binary relation of inclusion restricted to $P(X)$. Show for any infinite cardinal $\kappa$, that any $\kappa^+$-saturated model of $T$ has cardinality at least $2^\kappa$.

21. Let $\kappa$ be an infinite singular cardinal. Show that there is no linear order of cardinality $\kappa$ which is $\kappa$-saturated.

22. Prove that if $\mathcal{A} \equiv \mathcal{B}$ and both are $\omega$-saturated, then $\mathcal{A} \equiv_{\omega} \mathcal{B}$.

23. Prove that if $T$ is an $\omega$-categorical theory in a countable language, then all models of $T$ are $\omega$-saturated.

24. (Keisler-Morley) Let $M$ be a countable model of ZFC and suppose $M \models \ulcorner \kappa \text{ is a regular cardinal} \urcorner$. Show that there exists an elementary extension
$N$ of $M$ and $c \in N - M$ such that $N \models "c < \kappa"$ but for every $\alpha, \beta$ if $N \models "\alpha < \beta < \kappa"$ and $\beta \in M$, then $\alpha \in M$, i.e. the first new ordinal is just below $\kappa$.

In next seven problems below, let $T$ be a consistent complete theory in a countable language $L$.

25. Show that $T$ has a model $A$ such that for every $a, b \in A$; the type of $a$ in $A$ equals the type of $b$ in $A$ iff there exists a $L$-formula $\theta(x, y)$ such that $A \models \theta(a, b)$ and for all $L$-formula $\sigma(x)$:

$$T \vdash \theta(x, y) \rightarrow (\sigma(x) \iff \sigma(y))$$

26. (Ehrenfeucht) Give an example of a $T$ with exactly four nonisomorphic countable models.

27. (Ehrenfeucht) Give an example of a $T$ in a finite language with exactly three nonisomorphic countable models.

28. Suppose every model of $T$ is $\omega$-homogeneous. Show that if $T$ is not $\omega$-categorical, then $T$ has infinitely many nonisomorphic countable models.

29. If $T$ has two countable models such that neither can be elementarily embedded in the other, then show that $T$ has at least five nonisomorphic countable models.

30. A model $A$ is called minimal iff it has no proper elementary substructures. Suppose $T$ has a prime model and show the following are equivalent:

1. The prime model of $T$ is minimal.

2. Every atomic model of $T$ is countable.

31. Prove that $T$ has an $\omega$-homogeneous model of cardinality $\omega_1$ which has a countable type spectrum.

32. Let $T$ be a complete consistent theory in a countable language $L$. Let $L_C = L \cup \{c_n : n \in \omega\}$ where $c_n$ are new constant symbols. Show that there exists $\theta_s$ for $s \in 2^{< \omega}$ sentences in the language $L_C$ such that

1. $T \cup \{\theta_s\}$ is consistent;

2. if $s \subseteq t$, then $\vdash \theta_t \rightarrow \theta_s$;
3. if $x \in 2^\omega$, then $T \cup \{\theta_{xn} : n \in \omega\}$ is a complete consistent Henkin theory, with canonical model $A_x$; and

4. if $x \neq y \in 2^\omega$, then the only complete types realized in both $A_x$ and $A_y$ are principal.

33. Prove Shelah’s omitting types theorem: If $T$ is a complete consistent theory in a countable language $L$ and $\Sigma$ is a family of fewer than continuum many complete nonprincipal types over $T$, then $T$ has a model omitting all the types in $\Sigma$.

34. Let $(N, \in)$ be a countable transitive model of ZFC and let $(N^*, \in^*)$ be any countable elementary superstructure of $N$ with a nonstandard $\omega$. For any structure $A$ in $N$ let $A^*$ be the corresponding structure in $N^*$, i.e.

$$A^* = \{a \in N^* : N^* \models a \in A\}$$

for any relation symbol $R$

$$R^* = \{\bar{a} : N^* \models \bar{a} \in R^A\}$$

and similarly for operation symbols. Suppose $A, B \in N$ are structures in a countable language $L \in N$. Show that if $A \equiv B$ then the type spectrum of $A^*$ is equal to the type spectrum $B^*$ (Hint: use the fact that $\omega^*$ is nonstandard, and the fact that $N^*$ thinks that $A^*, B^*$ model all the same nonstandard sentences. It is also true that $A^* \equiv_{\omega^*} B^*$.) Suppose $A \in N$ is a structure in a countable language $L \in N$. Show that $A^*$ is $\omega-$homogeneous. Use this exercise to give another proof of the main lemma in Vaught’s two cardinal theorem. Namely if $T$ has a pair of models $A \preceq B$ such that $A \neq B$ but $U^A = U^B$, then $T$ has a pair of countable models $A \preceq B$ such that $A \neq B$, $U^A = U^B$, $A$ is $\omega-$homogeneous, and $A \simeq B$.

35. Prove the finite Ramsey theorem. For every $n, k, m \in \omega$ there exists arbitrarily large $l \in \omega$ such that $l \rightarrow (m)^k_n$, i.e. for every $f : [l]^k \rightarrow n$ there exists $H \in [l]^m$ such that $f | [H]^k$ is constant. (Hint: use compactness and the infinite version of Ramsey’s theorem.)

36. Prove that every infinite partially ordered set contains an infinite linear order or an infinite antichain. ( $X$ is an antichain iff any two elements of $X$ are incomparable, i.e. for every $a \neq b \in X$ ($\neg a \leq b$). (Hint: Ramsey’s theorem)
37. Let $A_i$ be an $L$ structures for each $i \in I$, $(X_i, \leq_i)$ a set of $A_i$ indiscernibles, and $U$ be an ultrafilter on $I$. Show that $\Pi X_i/U$ is a set of indiscernibles for $\Pi A_i/U$.

38. Let $A_n$ be infinite structures in a countable language $L$ and let $U$ be a nonprincipal ultrafilter on $\omega$. Show that $\Pi A_n/U$ contains a set of indiscernibles of cardinality the continuum. (Hint: Find $X_n \in [A_n]^\omega$ such that for every finite set of formulas $\Delta$ for all but finitely many $n \in \omega$ $X_n$ is a set of $\Delta$–indiscernibles in $A_n$.)

39. Do any of the problems 3.3.1–20 on p.153–155 of Chang and Keisler. I think the third sentence of 3.3.7 is false, can you give a counterexample?

40. Prove that $\kappa \nrightarrow \left(\omega\right)_2^\omega$. That is there exists $P_0 \cup P_1 \subset [\kappa]^\omega$ such that there does not exist $X \in [\kappa]^\omega$ with $[X]^\omega \subset P_0$ or $[X]^\omega \subset P_1$. Hint: first prove it for $\kappa = \omega$ then extend to arbitrary $\kappa$ by consider a maximal family of subsets $A \subset [\kappa]^\omega$ which are almost disjoint, i.e. $X, Y \in A$ implies $X \cap Y$ is finite.

41. (Vaught) Prove the following extension of Vaught’s two cardinal theorem for cardinals far apart. Let $T$ be a first order theory in a countable language which includes two unary predicate symbols $U, V$. We say that $A$ is a $(\kappa, \lambda, \delta)$-model of $T$ iff $|A| = \kappa$, $|V^A| = \lambda$, $|U^A| = \delta$. Prove that if for every $n \in \omega$ if $T$ has a $(\leq_n, \omega_1, \omega)$-model, then for every $\kappa \geq \omega_1$ $T$ has a $(\kappa, \omega_1, \omega)$-model. Hint: Assume wlog that $T$ has built in Skolem functions and $T \vdash U(x) \rightarrow V(x)$. Consider the theory $T^*$ which contains $T$ and also for every $L$-formula $\theta(\bar{x})$:

$$\forall \bar{x}[V(x_1) \land \ldots \land V(x_n)] \rightarrow [\theta(\bar{x}, \bar{a}) \iff \theta(\bar{x}, \bar{b})]$$

where $\bar{a}, \bar{b} \in I^{<\omega}$ have the same order type and $I = \{c_n : n \in \omega\}$.

42. (Vaught, Fuhrken) Show that the Hanf number of $L(Q)$ is $\kappa$ where $Qx$ means “there exists uncountably many $x$”.

Hint: Let $L$ be any first order language. Let $U, V$ be new unary predicate symbols and $F$ a new 3-ary relation symbol.

The idea is that $A \models \neg Qx\theta(x)$ iff $\{x : A \models \theta(x)\}$ is countable iff there exists a 1-1 function from $\{x : A \models \theta(x)\}$ to a fixed countable unary predicate $U$. Similarly $A \models Qx\theta(x)$ iff $\{x : A \models \theta(x)\}$ is uncountable iff there exists a 1-1 function from a fixed uncountable unary predicate $V$ to $\{x : A \models \theta(x)\}$.

For any $\theta^*$ a $L(Q)$-formula in prenex normal form define a first order $L \cup \{U, V, F\}$-formula $\theta^*$ as follows:
1. If $\theta$ is quantifier free then $\theta^* = \theta$;

2. $(\exists x \theta)^* = \exists x(\theta^*)$;

3. $(\forall x \theta)^* = \forall x(\theta^*)$;

4. $(Qx \theta)^* = \exists u \forall v [V(v) \rightarrow (\exists x F(u, v, x) \land \theta^*)]$;

5. $(\neg Qx \theta)^* = \exists u \forall x (\theta^* \rightarrow \exists v [F(u, x, v) \land U(v)]]$

In last two items $u, v$ are variables not occuring in $\theta^*$.

Let $\psi$ be the first order sentence which says that $F$ is the graph of a parameterized family of 1-1 functions, i.e. for each $u$ the set $\{(v, x) : F(u, v, x)\}$ is the graph of a partial 1-1 function.

Show that a $L(Q)$-sentence $\theta$ has a model of cardinality $\kappa$ iff $\theta^* \land \psi$ has a $(\kappa, \omega_1, \omega)$-model.

43. Use Ehrenfeucht games to show that for every sentence $\theta$ in the language of one binary relation $<$ there exists $n \in \omega$ such that $(\omega^n, <) \models \theta$ iff $(\omega^\omega, <) \models \theta$.

(This is ordinal exponentiation.)

44. Use Ehrenfeucht games to prove that

$$(\omega, <) \equiv (\omega + Z, <)$$

45. Any of the exercises 2.5.1-2.5.7.

46. (Lachlan) Let $T$ be a complete consistent theory in a countable language $L$. Show that for any $L$-formula $\theta(x)$ if rank($\theta(x)$) $\geq \omega_1$ then rank($\theta(x)$) $= \infty$. Hint: Look at a model of ZFC$^*$ with a standard $\omega$ but which thinks rank($\theta(x)$) $\geq \alpha$ for a nonstandard $\alpha < \omega_1$.

47. Let $\mathcal{A} = (A, E)$ be an equivalence relation. Show that Th($\mathcal{A}$) is $\omega$-stable. Compute rank($x = x$).

48. Let IND be the theory of infinitely many independent unary predicates, (example 3.4.2 p.157 Chang and Keisler). Show that IND is superstable but not $\omega$-stable.

49. Let $T$ be the theory of infinitely many equivalence relations $E_n$ for $n \in \omega$ such that each $E_n$ equivalence class is the union of infinitely many $E_{n+1}$ equivalence classes. Prove that $T$ is stable but not superstable.