Theorem 1 (Morley) Suppose $T$ is a theory in a countable language $L$ with built in Skolem functions and $< \Sigma_n : n \in \omega >$ is a countable family of types such that for every $\alpha < \omega_1$ $T$ has a model $A$ omitting each $\Sigma_n$ and $|A| = \beth_\alpha$. Then $T$ has a model with an infinite set of indiscernibles omitting each $\Sigma_n$.

Hence by the expanding technique of indiscernibles $T$ has arbitrarily large models omitting each $\Sigma_n$ and so the Hanf number of omitting types is $\beth_{\omega_1}$.

Lemma 2 Suppose $N$ is a model of $\text{ZFC}$ such that $\omega^N$ is standard but $\omega_1^N$ contains an infinite descending sequence. Suppose that $T$ and the $\Sigma_n$ are in $N$ and $N$ models:

"$T$ is a theory in a countable language $L$ with built in Skolem functions and $< \Sigma_n : n \in \omega >$ is a countable family of types such that for every $\alpha < \omega_1$ $T$ has a model $A$ omitting each $\Sigma_n$ and $|A| = \beth_\alpha$.”

Then $T$ has a model with an infinite set of indiscernibles omitting each $\Sigma_n$.

proof: For simplicity assume we just have a single 1-type, $\Sigma(x)$. Let $\alpha_n$ for $n \in \omega$ be such that $N \models \omega_1 > \alpha_n > \alpha_{n+1} + \omega$. Hence for every $n \in \omega$ and $k \in \omega$

$$N \models \beth_{\alpha_n} \rightarrow (\beth_{\alpha_{n+1}})^k$$

Let $N \models \text{A models } T, \text{ omits } \Sigma(x), |A| = \beth_{\alpha_0}$.” Build a sequence $X_n$ of elements of $N$ such that $N \models “X_{n+1} \subset X_n \subset A; |X_n| = \beth_{\alpha_n}”$ and also for every $L$-formula $\theta(x)$ there exists $n \in \omega$ such that

$$N \models \forall \bar{a} \in [X_n]^\omega A \models \theta(\bar{a})$$

or

$$N \models \forall \bar{a} \in [X_n]^\omega A \models \neg \theta(\bar{a})$$

and for every $L$-term $\tau(x)$ there exists $n \in \omega$ and $\theta_{\tau}(x) \in \Sigma$ such that

$$N \models \forall \bar{a} \in [X_n]^\omega A \models \neg \theta_{\tau}(\bar{a})$$
Let $I = \{c_n : n \in \omega\}$ and consider the theory:

$$T \cup \{\theta(\vec{a}) \iff \theta(\vec{b}) : \vec{a}, \vec{b} \in I^{<\omega}\} \cup \{-\theta_{\tau}(\vec{a}) : \vec{a} \in I^{<\omega}\}$$

Each finite subset of this is consistent and it gives an E-M type which does the job.

**Lemma 3** Suppose $M$ is a countable model of ZFC. Then there exists a model $N$ such that $M \preceq N$ and

$$\{n \in M : M \models n \in \omega\} = \{n \in N : N \models n \in \omega\}$$

but $\omega_1^N$ contains an infinite descending sequence.

**proof:** Let

$$T = D_{\text{elem}}(M) \cup \{c_0 < \omega^M_1, c_{n+1} < c_n : n \in \omega\} \cup \{c_n > \alpha : \alpha < \omega^M_1, n \in \omega\}$$

and let

$$\Sigma(x) = \{x < \omega^M\} \cup \{x \neq n : n < \omega^M\}$$

By the omitting types theorem it is enough to show that $\Sigma$ is not isolated over $T$.

**Claim** For any sentence $\theta(\vec{c})$

$$T \cup \{\theta(c_0, c_1, \ldots, c_n)\} \text{ is consistent}$$

iff

$M$ models:

$$\forall \alpha < \omega_1 \exists x_0, x_1, \ldots, x_n[(\alpha < x_n < \ldots < x_1 < x_0 < \omega_1) \land (\theta(x_0, x_1, \ldots, x_n))]$$

The proof is left to reader. Abbreviate the last statement:

$$\exists^{cof\omega_1} \vec{x} \theta(\vec{x})$$

Suppose that $\psi(x, \vec{c})$ isolates the type $\Sigma(x)$. Hence $T \models \psi(x, \vec{c}) \rightarrow x < \omega$. By the claim

$$M \models \exists^{cof\omega_1} \vec{y} \exists x < \omega \psi(x, \vec{y})$$
Since $\omega_1$ is uncountable $\exists n \in M$ such that

$$M \models n < \omega \land \exists^{\text{cof}\omega_1} y \psi(n, \vec{y})$$

But then $T \cup \{\psi(n, \vec{c})\}$ is inconsistent, contradiction.

To prove Morley’s theorem, first note that both lemmas are true for any sufficiently nice fragment of ZFC. For example let $\text{ZFC}^*$ be the theory of $(V_\kappa, \in)$ where $\kappa$ is any regular cardinal greater than $\beth_{\omega_1}$. So given any $T$ a theory in a countable language $L$ with built in Skolem functions and $< \Sigma_n : n \in \omega >$ a countable family of types such that for every $\alpha < \omega_1$ $T$ has a model $A$ omitting each $\Sigma_n$ and $|A| = \beth_\alpha$, use the Lowenheim-Skolem theorem to find a countable model $M \preceq V_\kappa$ which contains $T$ and $< \Sigma_n : n \in \omega >$. Apply Lemma 3 to get a model $N$ and then apply Lemma 2 to get the conclusion of the theorem for $T$ and $< \Sigma_n : n \in \omega >$. 

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