GENERIC SOUSLIN SETS

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By iterated forcing we create generic Souslin sets, which we use to answer questions of Ulam, Hansell, and Mauldin. For $X$ a topological space a set $Y \subseteq X$ is analytic in $X$ (also called Souslin in $X$ or $\Sigma_1^1$ in $X$) iff there are Borel sets $B_s$ for $s \in \omega^{<\omega}$ such that:

$$Y = \bigcup_{f \in \omega^{<\omega}} \bigcap_{n < \omega} B_{f \upharpoonright n}.$$ 

For $X = 2^\omega$ (the Cantor space) a set $Y \subseteq X$ is analytic iff it is the projection of a Borel subset of $2^\omega \times 2^\omega$. Given $R \subseteq P(X)$ (the power set of $X$) let $B(R)$ be the smallest family of subsets of $X$ including $R$ and closed under countable union and complementation (i.e., the $\sigma$-algebra generated by $R$). If $X$ is a topological space and $R$ the family of open sets then $B(R)$ is the family of Borel subsets of $X$. The following question was raised by Ulam.

1. Does there exist $R \subseteq P(2^\omega)$ such that $R$ is countable and every analytic set in $2^\omega$ is an element of $B(R)$?

Rothberger showed that assuming CH there is such a $R$. We will show that it is consistent with ZFC that there is no such $R$.

2. Does there exist a separable metric space $X$ in which every subset is analytic but not every subset is Borel?

This was raised by R. W. Hansell. Clearly CH implies no such $X$ exists. We show that it is consistent with ZFC that such a $X$ exists.

Let $R = \{ A \times B : A, B \subseteq 2^\omega \}$, the abstract rectangles in the plane. Let $S(R)$ be the family of subsets of $2^\omega \times 2^\omega$ obtained by applying the Souslin operation to sets in $B(R)$. The next question was asked by D. Mauldin.

3. Does $S(R) = P(2^\omega \times 2^\omega)$ imply $B(R) = P(2^\omega \times 2^\omega)$?

We show that the answer to this question is no.

Preliminaries. Recall the following definitions:

1. $\omega = \{ 0, 1, 2, \cdots \}$ and $\forall n < \omega, n = \{ m \mid m < n \}$;
2. $\omega^n = \{ s \mid s : n \to \omega \}$;
3. for $s \in \omega^n$ and $n < \omega$, $s \upharpoonright n$ is that $t \in \omega^{n+1}$ such that $t \upharpoonright m = s$ and $t(m) = n$;
4. $\phi$ denotes the empty sequence;
5. $\omega^{<\omega} = \cup \{ \omega^n : n < \omega \}$;
6. $T \subseteq \omega^{<\omega}$ is a tree iff $\forall s, t \in \omega^{<\omega} (s \subseteq t \in T \to s \in T)$;
7. $T$ is a well founded tree iff $\forall f \in \omega^n \exists n < \omega f \upharpoonright n \in T$;
8. for $s \in T$ a well founded tree $|s|_T$ is defined inductively by:
\[ |s|_\tau = \sup \{ |s^n|_\tau + 1 : \exists n \ s^n \in T \} ; \]

(9) for \( \alpha < \omega_1 \), \( T \) is a normal \( \alpha \)-tree iff
(a) \( T \) is a well founded tree such that \( |\phi|_\tau = \alpha \);
(b) if \( s \in T \) and \( |s|_\tau > 0 \), then \( \forall n \ s^n \in T \);
(c) if \( s \in T \) and \( |s|_\tau = \beta + 1 \), then \( \forall n \ |s^n|_\tau = \beta \); 
(d) if \( s \in T \) and \( |s|_\tau = \lambda \) where \( \lambda \) is a limit ordinal, then \( \forall \beta < \lambda \), \( \{ n : |s^n|_\tau < \beta \} \) is finite (see [9]);

(10) for \( T \subseteq \omega^{<\omega} \) a tree define:
\[ P(T) = \{ p \mid \exists F \in [T]^{<\omega}, p : F \rightarrow 2, \forall s < \omega, \forall s \in \omega^{<\omega} \text{ if } s, s^n \in F \text{, then } p(s) = 1 \implies p(s^n) = 0 \} \]
\( P(T) \) is ordered by inclusion.

(11) A notion of rank on a partial order \( P \) is a function whose domain is a subset of \( P \) and whose range is the ordinals. For \( \alpha \) an ordinal and \( p \in P \), we let \( |p| = \alpha \) mean that \( p \) is in the domain of this function and its value is \( \alpha \). The following property must be satisfied. For every \( p \in P \) and \( \beta \geq 1 \), there exists \( \hat{p} \in P \) compatible with \( p \) such that \( |\hat{p}| \leq \beta \) and for every \( q \in P \) if \( |q| < \beta \) and \( \hat{p} \) and \( q \) are compatible, then \( p \) and \( q \) are compatible.

(12) Given a notion of rank on \( P \) if \( \tau \) is a term such that \( \models "\tau \in \omega^{<\omega}" \), then we say that \( |\tau| = 0 \) iff for any \( p \in P \) and \( n < \omega \) there exists \( q \in P \) compatible with \( p \) such that \( |q| = 0 \) and \( s \in 2^s \) such that \( q \models "s \subseteq \tau" \).

(13) For \( T \) a normal \( \alpha \)-tree and \( p \in P(T) \) define \( |p| \) to be the maximum \( |s|_\tau \) for \( s \in \text{dom}(p) \).

(14) \( T^* = \{ s \in T : |s|_\tau = 0 \} \).

The following lemma is key. It implies that \( |p| \) is a rank on \( P(T) \).

**LEMMA 1.** \( \forall \beta \geq 1 \forall p \in P(T) \exists \hat{p} \in P(T) \) such that
(a) \( p \) and \( \hat{p} \) are compatible;
(b) \( p \upharpoonright T^* = \hat{p} \upharpoonright T^* \);
(c) \( |\hat{p}| \leq \beta \);
(d) \( \forall q \in P(T) \) if \( |q| < \beta \), then \( \hat{p} \) and \( q \) are compatible implies \( p \) and \( q \) are compatible.

**Proof.** This is essentially Lemma 2 of [10]. We reprove it here for completeness. Let \( F = \{ s^n : s \in \text{dom}(p) \} \), \( p(s) = 1 \), \( |s|_\tau = \lambda \) a limit ordinal \( > \beta \), and \( |s^n|_\tau < \beta \). By normality of \( T \), \( F \) is finite, and \( \forall t \in F \), \( |t|_\tau \geq 2 \). Thus we can find \( r \supseteq p \forall t \in F \exists m \ t^m \in \text{dom}(r) \) and \( r(t^m) = 1 \). Let \( D = \{ s \in \text{dom}(r) : |s|_\tau \leq \beta \} \) and \( \hat{p} = r \upharpoonright D \). \( \hat{p} \) and \( \hat{p} \) are compatible since \( r \) extends them both. \( p \upharpoonright T^* = \hat{p} \upharpoonright T^* \) since \( \forall t \in F \forall m \ t^m \upharpoonright |t|_\tau \geq 1 \).

Now we check (d). Suppose \( |q| < \beta \) and \( p \) and \( q \) are not compatible. Then there are \( s \in \text{dom}(p) \) and \( t \in \text{dom}(q) \) which demonstrate
that \( p \cup q \) is not a condition.

Case 1. \( s = t \) and \( p(s) \neq q(t) \). Since \( |q| < \beta \) it follows \( |t|_\tau < \beta \) and so \( s \in \text{dom}(\hat{p}) \).

Case 2. \( s = t \cdot m \) for some \( m \) and \( p(s) = q(t) = 1 \). But then \( |s|_\tau < |t|_\tau < \beta \) and so again \( s \in \text{dom}(\hat{p}) \).

Case 3. \( t = s \cdot m \) for some \( m \) and \( p(s) = q(t) = 1 \). Since \( |t|_\tau < \beta \) either \( |s|_\tau \leq \beta \) and so \( s \in \text{dom}(\hat{p}) \) or \( |s|_\tau = \lambda \) a limit ordinal > \( \beta \) in which case \( t \in F \) so there exists \( n < \omega \) such that \( \tau(t \cdot n) = 1 \) and so \( t \cdot n \in \text{dom}(\hat{p}) \) and so \( \hat{p} \) and \( q \) are incompatible. In all three cases \( \hat{p} \) and \( q \) are incompatible. \( \square \)

The next lemma asserts the fact that statements of small rank should be forced by conditions of small rank. \( M \) is the ground model of ZFC and \( P \) is any partial order with a notion of rank.

Lemma 2. Let \( B(\tau) \) be any \( \Sigma^0_\beta \) predicate with parameter in \( M \), \( 1 \leq \beta \), \( \Vdash \neg \psi \tau \in 2^\omega \), \( |\tau| = 0 \), and \( p \in P \) such that \( p \Vdash \neg \langle B(\tau) \rangle \). Then \( \exists \hat{p} \in P, |\hat{p}| < \beta, p \) and \( \hat{p} \) are compatible and \( \hat{p} \Vdash \langle B(\tau) \rangle \).

Proof. The proof is by induction \( \beta \).

Case 1. \( \beta = 1 \). Then \( p \Vdash \langle \exists n R(\tau \upharpoonright n, x \upharpoonright n) \rangle \) where \( R \) is primitive recursive and \( x \in M \cap 2^\omega \). Find \( q \) extending \( p \) and \( s \in 2^\omega \) for some \( n \) such that \( q \Vdash \langle \tau \upharpoonright n = s \rangle \) and \( R(s, x \upharpoonright n) \) holds. By the definition of \( |\tau| = 0 \), \( \exists \hat{p} \) compatible with \( q \) (and hence with \( p \)) such that \( |\hat{p}| = 0 \) and \( \hat{p} \Vdash \langle \tau \upharpoonright n = s \rangle \). Thus \( \hat{p} \Vdash \langle \exists n R(\tau \upharpoonright n, x \upharpoonright n) \rangle \).

Case 2. \( \beta \) a limit ordinal. Then \( p \Vdash \langle \exists n B_n(\tau) \rangle \) where each \( B_n(\tau) \) is a \( \Sigma^0_n \) predicate for some \( \beta_n < \beta \). Let \( p_0 \) extend \( p \) such that \( \exists n < \omega p_0 \Vdash \langle B_n(\tau) \rangle \). By induction \( \exists \hat{p} \) compatible with \( p_0 \) (and hence with \( p \)) such that \( |\hat{p}| < \beta_n \) and \( \hat{p} \Vdash \langle B_n(\tau) \rangle \) (and hence \( \hat{p} \Vdash \langle \exists n B_n(\tau) \rangle \)).

Case 3. \( \beta = \gamma + 1 \) and \( \gamma > 0 \). As in Case 2 we may as well assume \( p \Vdash \langle \neg B(\tau) \rangle \) where \( B(\tau) \) is a \( \Pi^0_\gamma \) predicate. By Lemma 1, \( \exists \hat{p} \in P, \hat{p} \) and \( p \) compatible, \( |\hat{p}| \leq \gamma \), and \( \forall q \in P \) if \( |q| < \gamma \) and \( q \) and \( \hat{p} \) are compatible, then \( q \) and \( p \) are compatible. Then \( \hat{p} \Vdash \langle \neg B(\tau) \rangle \). Otherwise \( \exists r \) extending \( \hat{p} \), \( r \Vdash \langle \neg B(\tau) \rangle \). Since \( \neg B(\tau) \) is a \( \Sigma^0_\gamma \) predicate, by induction \( \exists \hat{r} \in P, |\hat{r}| < \gamma, \hat{r} \) and \( r \) compatible, and \( \hat{r} \Vdash \langle \neg B(\tau) \rangle \). But \( \hat{r} \) and \( p \) are incompatible (since \( p \Vdash \langle B(\tau) \rangle \)) and so by choice of
\(\hat{p}, \hat{r}\) and \(\hat{p}\) are incompatible a contradiction.

Next we describe almost disjoint forcing (similar to the way it is done in [2]). Given \(X = \{x_\alpha: \alpha < \omega_1\} \subseteq 2^\omega\) distinct and \(\langle Y_\alpha: \alpha < \omega_1\rangle = Y\) where each \(Y_\alpha \subseteq \omega^{<\omega}\), we want to force a sequence of \(G_\alpha\) sets \(\langle G_\alpha: s \in \omega^{<\omega}\rangle\) such that \(\forall s \forall \alpha (x_\alpha \in G_\alpha \iff s \in Y_\alpha)\). Let \(B\) be the family of all clopen subsets of \(2^\omega\). Define \(P(X, Y)\) as follows:

- it is the set of all \(r\) such that
  - (a) \(r\) is a finite subset of \(\omega^{<\omega} \times \omega \times (B \cup X)\);
  - (b) if \(\langle s, n, B\rangle, \langle s, n, x_\alpha\rangle \in r\) then \(x_\alpha \not\in B\);
  - (c) if \(\langle s, n, x_\alpha\rangle \in r\) then \(s \in Y_\alpha\).

As usual \(r\) extends \(p\), \((r \supseteq p)\) iff \(r \supseteq p\). It is well known that \(P(X, Y)\) satisfies the c.c.c. and also for any \(G\) which is \(P(X, Y)\)-generic if we define \(G_\alpha = \bigcap_n \cup \{B: \langle s, n, B\rangle \in G\}\) then \(\forall s \forall \alpha (x_\alpha \in G_\alpha \iff s \in Y_\alpha)\).

1. Forcing a Souslin set. We now describe how to force Souslin sets. Let \(M\) be our ground model of ZFC. Working in \(M\) let \(F^*\) be some standard fixed bijection between \(\omega^{<\omega}\) and \(\omega\), and define \(F: 2^\omega \to 2^{\omega^{<\omega}}\) by \(F(x)(s) = x(F^*(s))\). Let \(X = \{x_\alpha: \alpha < \omega_1\}\) be a fixed subset of \(2^\omega\) such that for all \(\alpha < \omega_1\), \(F(x_\alpha)\) is the characteristic function of a normal \(\alpha\)-tree \(T_\alpha\). Let

\[P_0 = \sum_{\alpha < \omega_1} P(T_\alpha),\]

note that \(P_0\) has c.c.c. since it is equivalent to adding \(\omega_1\) Cohen reals. Note that any \(G\) which is \(P(T_\alpha)\)-generic over \(M\) determines (and is determined by) a map \(G_\alpha: T_\alpha \to 2\). \(G_\alpha \upharpoonright T^*_\alpha\) in fact determines \(G_\alpha\) by the rule \(G_\alpha(s) = 1\) iff \(\forall n G_\alpha(s^n) = 0\). Given \(G^*_0\) \(P_0\)-generic over the ground model \(M\), let \(G^* = \langle G_\alpha: \alpha < \omega_1\rangle\) and let \(y_\alpha = \{s \in T^*_\alpha: G_\alpha(s) = 0\}\). Let \(P_1 = P(X, Y)\) where \(Y = \langle y_\alpha: \alpha < \omega_1\rangle\). (So \(P_1 \in M[G^0]\).) Let \(P = P_0 P_1\).

Working in \(M[G]\) for \(G\) \(P\)-generic over \(M\) (so \(G = \langle G_\alpha: \alpha < \omega_1\rangle, \langle G^*_\alpha: s \in \omega^{<\omega}\rangle\)) let:

\[A = \{x_\alpha \in X: G_\alpha(\phi) = 1\} .\]

To see that \(A\) is analytic in \(X\) we will define \(\hat{A}\) a \(\Sigma^1_1\) set such that \(\hat{A} \cap X = A\). Define \(x \in \hat{A}\) iff \(\exists T \subseteq \omega^{<\omega}, \exists p: \omega^{<\omega} \to 2, \exists T^* \subseteq \omega^{<\omega}\) such that

- (a) \(F(x)\) is the characteristic function of \(T\);
- (b) \(T\) is a tree;
- (c) \(T^* = \{s \in T: \forall n \ s \upharpoonright n \in T\} = \{s \in T: \forall n \ s \upharpoonright n \not\in T\}\);
- (d) \(\forall s \in T^* \ p(s) = 1\) iff \(x \in G_s\);
- (e) \(\forall s \in T - T^* \ p(s) = 1\) iff \(\forall n \ p(s \upharpoonright n) = 0\);
- (f) \(p(\phi) = 1\).
(a) thru (f) are easily seen to be a Borel predicate of $x, T, T^*$, and $p$, and hence $A$ is $\Sigma^1_1$.

In order to show $A$ is a new Souslin set we first want to extend our notion of rank to $P$. Let $Q = \{ r : r$ satisfies (a) and (b) in the definition of $P(X, Y) \}$ (thus $Q \in M$). Then

$$\{(p, q) : p \in P_\alpha, q \in Q, \text{ and } p \vdash "q \in P(X, Y)"\}$$

ordered by $(\hat{p}, \hat{q}) \geq (p, q)$ iff $\hat{p} \geq p$ and $\hat{q} \geq q$, is clearly dense in $P$, so for simplicity assume it is $P$. Let us unravel $p \vdash "q \in P(X, Y)"$. This means that whenever $\langle s, n, x_\alpha \rangle \in q'$ then $p \vdash "s \in Y_\alpha"$. But $p \vdash "s \in Y_\alpha"$ iff $s \in T_\alpha^*$ or $(s \in T_\alpha^*, s \in \text{dom}(p_\alpha), \text{ and } p_\alpha(s) = 1)$. The fact which we note is that if $p, p' \in P_\alpha$ and $\forall \alpha < \omega_1, p_\alpha \upharpoonright T_\alpha^* = p'_\alpha \upharpoonright T_\alpha^*$, then $\forall \alpha < \omega_1, p_\alpha \upharpoonright T_\alpha^*$ and $q$ are compatible. Let $p_\alpha$ be defined by:

$$|p_\alpha|_\alpha = \max \{|s|_\tau : \gamma \geq \alpha \text{ and } s \in \text{dom}(p_\tau)\}.$$  

Note that the rank depends only on the part of the condition in $P_\alpha$. To see that it is a rank function, let $(p, q)$ be any condition and $\beta \geq 1$. For each $\gamma > \alpha$ by Lemma 1 $\exists \hat{p}_\gamma \in P(T_\gamma)$ such that $\hat{p}_\gamma \upharpoonright T_\gamma^* = p \upharpoonright T_\gamma^*$, $\hat{p}_\gamma$ and $p$ are compatible, $|\hat{p}_\gamma| \leq \beta$, and $\forall q \in P(T_\gamma)$ if $|q| < \beta$ and $\hat{p}_\gamma$ and $q$ are compatible, then $p_\gamma$ and $q$ are compatible. Let $\hat{p} \in P_\alpha$ be defined by:

$$\hat{p}_\gamma = \begin{cases} p_\gamma & \text{if } \gamma \leq \alpha \\ \hat{p}_\gamma & \text{if } \gamma > \alpha. \end{cases}$$

By what we have already remarked

$$(\hat{p}, q) \in P, |(\hat{p}, q)|_\alpha \leq \beta, (p, q) \text{ and } (\hat{p}, q) \text{ are compatible},$$

$$\forall (p', q') \in P \text{ if } |(p', q')|_\alpha < \beta \text{ and } (p', q') \text{ is compatible with } (\hat{p}, q), \text{ then } (p', q') \text{ is compatible with } (p, q).$$

Let $G$ be $P$-generic over $M$, and let $A$ be the generic Souslin subset of $X$ determined by $G$. We first show that $M[G] \models "A \text{ is not Borel in } X"$. Suppose on the contrary that $\exists \tau, wB(\nu, \omega)\alpha\Sigma^1_1$ predicate with parameters in $M$, and $r \in P$ such that

$$r \vdash "\forall x \in X(x \in A \text{ if } B(\tau, x))".$$  

By c.c.c. we can find $\alpha < \omega_1$ such that $|\tau|_\alpha = 0, |r|_\alpha = 0$, and $\beta < \alpha$. Let $\gamma$ be any countable ordinal greater than $\alpha + \omega$. Extend $r = (p, q)$ by adding $p_\gamma(\phi) = 1$ to $p$, and call the result $r_\gamma$. By this addition, $r_\gamma \vdash "x \in A"$, so $r_\gamma \vdash "B(\tau, x_\gamma)"$, so there exists $r_\gamma$ compatible with $r_\gamma$ such that $|r_\gamma|_\alpha < \beta$ and $r_\gamma \vdash "B(\tau, x_\gamma)"$. But since $\gamma > \alpha + \omega$ and $|r_\gamma|_\alpha < \beta < \alpha$, it follows that $\exists r_\gamma \geq r_\gamma$ such that $p_\gamma(\phi) = 0$ and
thus $\models "x_\gamma \notin A"$. This is a contradiction since $r_3$ and $r_1$ are compatible (since $r_2$ and $r_1$ are compatible).

Now let us prove something a little stronger. Let $M \models "H \subseteq P(X), |H| \leq \omega"$, then, we claim $M[G] \models "\forall A \in B(H) \text{ the } \sigma\text{-algebra generated by } H"$.

Work in $M$. Let $H = \{A_n: n < \omega\}$ and define $K: X \to 2^\omega$ by $K(x)(n) = 1$ iff $x \in A_n$. Let $Y$ be the range of $K$, then $K$ has the property that it maps the $\sigma$-algebra generated by $H$ into the Borel subsets of $Y$.

For any $C \in B(H)^{\omega[\alpha]}$, $\exists B$ Borel subset of $Y$, and $p \in P$ such that

$$p \models "\forall x \in X (x \in C \text{ iff } K(x) \in B)".$$  

The preceding proof now goes through. Finally we are ready to state the theorem.

**Theorem 3.** It is consistent with ZFC that there does not exist $H \subseteq P(2^\omega)$ countable such that every analytic set is in the $\sigma$-algebra generated by $H$.

**Proof.** Let $M, X$, and $P$ be as above. Working in $M$ let $\{P_\alpha: \alpha < \omega_2^e\}$ be a set of isomorphic copies of $P$. Force with $\Sigma\{P_\alpha: \alpha < \omega_2^e\}$. Let $\langle G_\alpha, \alpha < \omega_2^e \rangle$ be generic over $M$. If $M[G_\alpha: \alpha < \omega_2^e] \models "H \subseteq P(2^\omega), |H| \leq \omega"$ then by c.c.c. $\exists \alpha_0 < \omega_2^e$ such that $\{B \cap X: B \in H\} \in M[G_\alpha: \alpha \neq \alpha_0]$. Let $M[G_\alpha: \alpha \neq \alpha_0]$ be the new ground model and $\hat{A}$ the analytic set created by $P_{\alpha_0}$. Note that although $P_{\alpha_0}$ is not the same as adding Cohen reals, because of its finite nature it is the same partial order whether defined in $M$ or any extension of $M$ (e.g., $M[G_\alpha: \alpha \neq \alpha_0]$). We have already noted that $\hat{A} \cap X$ is not in the $\sigma$-algebra generated by $\{B \cap X: B \in H\}$ and therefore $\hat{A}$ is not in the $\sigma$-algebra generated by $H$. \qed

2. Making subsets generic Souslin sets. Let $\Sigma$ be the set of countable successor ordinals greater than two. As in §1 let $X^* = \{x_\alpha: \alpha \in \Sigma\} \subseteq 2^\omega$ and $F: 2^\omega \to 2^{(\omega < \omega)}$ be the map such that $\forall \alpha \in \Sigma, F(x_\alpha)$ is a normal $\alpha$-tree $T_\alpha$. For $i = 0$ or $1$ and $T \subseteq \omega^{<\omega}$ define:

$$P^i(T) = \{p \in P(T): \exists \hat{p} \text{ an extension of } p, \hat{p}(\phi) = i\}.$$  

It is easy to check that for any $G$ which is $P^i(T)$-generic over $M$, $G(\phi) = i$. Given $Z \subseteq \Sigma$ define $P(Z)$ a suborder of $P$ by $(p, q) \in P(Z)$ iff $(p, q) \in P$ and $\forall \alpha \in \Sigma$:

(a) if $\alpha \in Z$ then $p_\alpha \in P^i(T_\alpha)$;
(b) if $\alpha \notin Z$ then $p_\alpha \in P^i(T_\alpha)$.

As before for $G$ $P(Z)$-generic over $M$, in $M[G]$, $\{x_\alpha: \alpha \in Z\}$ is
analytic in $X^*$. The reason for $\Sigma$ will be evident in the proof of Lemma 5.

**Theorem 4.** There exist a generic extension $N$ of $M$ such that $N \models "\text{Every subset of } X^* \text{ is analytic in } X^* \text{ but some subset of } X^* \text{ is not Borel in } X^"."$

**Proof.** $N$ will be obtained by iterating with finite support $P(Z)$. Since each $P(Z)$ is a relatively simple suborder of $P$ we can give the following simpler definition. We assume $M \models "\omega_1 = \omega_2."$ Let $Q = \Sigma_{\alpha < \omega_2} P_\alpha$ as in §1 and for $p \in Q$ define $\text{supp}(p) = \{\alpha \in \omega_2 : p(\alpha) \neq 0\}$. Let $A_\alpha$ for $\alpha < \omega_2$ list with $\omega_2$ repetitions all maps $A: \omega_1 \to [Q]^\omega$. Inductively define $Q_\alpha \subseteq Q$ for $\alpha < \omega_2$. For $\alpha = 0$ let $Q_0 = \{p \in Q : \text{supp}(p) = 0\}$ (i.e., $Q_0 = P$). For all $\alpha Q_\alpha \subseteq \{p \in Q : \text{supp}(p) \subseteq \alpha\}$. For $\alpha$ a limit ordinal let $Q_\alpha = \bigcup \{Q_\beta : \beta \leq \alpha\}$. For $\alpha + 1$ let $G_\alpha$ be $Q_\alpha$-generic over $M$ and let $Z_\alpha = \{\beta \in \Sigma : A_\alpha(\beta) \cap G_\alpha \neq \emptyset\}$. Then

$$Q_{\alpha+1} = \{p \in Q \mid p \upharpoonright \alpha \in Q_\alpha, p \upharpoonright \alpha \equiv_{\alpha} "p(\alpha) \in P(Z_\alpha)"$$

and $\text{supp}(p) \subseteq \alpha + 1$.

(Of course by $p \upharpoonright \alpha$ here we mean that condition in $Q$ whose restriction to $\alpha$ is the same as $p$'s and whose support is contained in $\alpha$.)

Thus if $G_{\omega_2}$ is $Q_{\omega_2}$ generic over $M$ then $M[G_{\omega_2}] \models "\text{Every subset of } X^* \text{ is analytic in } X^"."$ Work in $M$. Given $\alpha < \omega_1$ recall the definition $|p|_\alpha$ for $p \in P$ given in §1. Given $K \subseteq \omega_2$ and $\alpha < \omega_1$, define a map $F: Q_{\omega_2} \to \alpha \cup \{\infty\}$ by $F(p) = \max\{|p(\delta)|_\alpha : \delta \in K\}$ if $\text{supp}(p) \subseteq K$ and the max is less than $\alpha$, and otherwise let $F(p) = \infty$. Denote $F(p)$ by $|p|_{(K, \alpha)}$. For suitably chosen $K$ and $\alpha$ we will show $|p|_{(K, \alpha)}$ is a rank function. Given $\Gamma \subseteq Q_{\omega_2}$ and $\theta$ a sentence we say $\Gamma$ decides $\theta$ if $\forall p \in Q_{\omega_2} \exists \hat{q} \in \Gamma p$ and $q$ are compatible, and $q \models "\theta"$ or $q \not\models "\theta"$.

**Lemma 5.** Suppose that $\forall \delta \in K \forall \beta < \alpha \{p \in Q_\delta : |p|_{(K, \alpha)} = 0\}$ decides "$\beta \in Z\alpha"". Then $|p|_{(K, \alpha)}$ is a rank function.

**Proof.** We must show that given $p \in Q_{\omega_2}$ and $1 \leq \beta \leq \alpha$ there exists $\hat{p} \in Q_{\omega_2}$ compatible with $p$, $|\hat{p}|_{(K, \alpha)} \leq \beta$, and $\forall q \in Q_{\omega_2}$ if $|q|_{(K, \alpha)} < \beta$ and $\hat{p}$ and $q$ are compatible, then $p$ and $q$ are compatible.

Recall that in the proof that $| |_\alpha$ is a rank function on $P$ we obtained for each $p \in P$ a $\hat{p} \in P$ such that:

(a) $|\hat{p}|_\alpha \leq \beta$;
(b) $\hat{p}$ and $p$ are compatible;
(c) $\forall q \in P$ if $|q|_\alpha < \beta$ and $q$ and $\hat{p}$ are compatible, then $q$ and $p$ are compatible;
(d) \( \forall \gamma < \alpha, \hat{p}(\gamma) = p(\gamma) \).

Given \( p \in Q_{\omega_2} \) define \( \hat{p} \) by letting \( \forall \delta \in K, \hat{p}(\delta) = 0 \) and \( \forall \delta \in K, \hat{p}(\delta) \) is the condition in \( P \) obtained above for \( p(\delta) \). We show that \( \hat{p} \in Q_{\omega_2} \).

Suppose not and let \( \delta \) be the least such that \( \hat{p} \nmid \delta \) does not force "\( p(\delta) \in P(Z^\delta) \)". Clearly \( \delta \in K \). Let \( p(\delta) = (p', q) \). Then there must be some \( \gamma \in \Sigma \) such that \( p'_\gamma \in P^0(T_\gamma) \) or \( p'_\gamma \in P^\gamma(T_\gamma) \), and \( \hat{p} \nmid \delta \) does not force "\( \gamma \in Z^\delta \)" respectively "\( \gamma \in Z^\delta \)". If \( p'_\gamma \in P^\gamma(T_\gamma) \) then \( \phi \in \text{dom}(p') \) and \( p'_\gamma(\phi) = 1 \). If \( p'_\gamma \in P^0(T_\gamma) \) then either \( \phi \in \text{dom}(p') \) and \( p'_\gamma(\phi) = 0 \) or \( \exists n < \omega, \langle n \rangle \in \text{dom}(p') \) and \( p'_\gamma(\langle n \rangle) = 1 \). Since \( \gamma \in \Sigma \) it is a successor ordinal. Since \( |p(\delta)|_\alpha \leq \beta < \alpha \) and \( |\langle n \rangle|_\tau, \geq \gamma - 1 \) it must be that \( \gamma < \alpha \). By the properties of \( K \) and \( \alpha, \exists q \in Q_{\omega_1} \) \( |q|_{(K, \alpha)} = 0, q \vdash \"\gamma \in Z^\delta \" \) (respectively "\( \gamma \in Z^\delta \" \), and \( q \) is compatible with \( \hat{p} \nmid \delta \). But since \( q \) is compatible with \( \hat{p} \nmid \delta \), it is compatible with \( p \nmid \delta \). This is a contradiction, since by (d) \( q \vdash \"\hat{p}(\delta) \in P(Z^\delta)\" \).

If \( A \) is the analytic subset of \( X^* \) which is created at the first step, then \( A \) is not Borel in \( X^* \) in the model \( M[G_{\omega_2}] \). To see this suppose not and \( \exists p \in Q_{\omega_2} \)

\[
p \vdash \"\forall \xi \in X^*(x \in A \iff x \in B.)\"
\]

where \( B \) is a \( \Sigma_3 \) set with parameter \( \tau \in 2^\omega \). Using the c.c.c. of \( Q_{\omega_2} \) it is easy to obtain \( K \subseteq \omega_1 \) countable, \( 0 \in K \), and \( \alpha < \omega_1 \) with \( \beta < \alpha \), such that \( |p|_{(K, \alpha)} = 0, |\tau|_{(K, \alpha)} = 0 \), and \( K \) and \( \alpha \) satisfy the requirements set down in Lemma 5. As in §1 this leads to a contradiction.

3. Abstract Souslin sets. Recall that \( R = \{A \times B: A, B \subseteq 2^\omega \} \), \( B(R) \) is the \( \sigma \)-algebra generated by \( R \), and \( S(R) \) the family of sets which are gotten by applying the Souslin operation to sets in \( B(R) \).

**Theorem 6.** It is consistent with ZFC that \( S(R) = P(2^\omega \times 2^\omega) \neq B(R) \).

The model used will be a minor modification of the one obtained in §2.

**Lemma 7.** Suppose \( X \subseteq 2^\omega, |X| = |2^\omega| \), and every subset of \( X \) of cardinality less than \( |2^\omega| \) is analytic in \( X \). Then \( S(R) = P(2^\omega \times 2^\omega) \).

**Proof.** Let \( \kappa = |2^\omega| \) and \( X = \{x_\alpha: \alpha < \kappa \} \). Since \( S(R) \) is closed under finite union, it is enough to show that any \( Y \subseteq \kappa \) with the property that \( (\alpha, \beta) \in Y \rightarrow \alpha \leq \beta \), is in \( S(R) \). For each \( \beta \) let \( X_\beta = \{x_\alpha: (\alpha, \beta) \in Y \} \). For each \( \beta \) and \( s \in \omega^\omega \) let \( C_s^\beta \) be a closed subset of \( X \) such that \( X_\beta = \bigcup_{s \in \omega^\omega} \bigcap_{n < \omega} C_s^\beta \bigcap_{n} \).
For each \( s \in \omega^<\omega \) define \( B_s = \{ \langle \alpha, \beta \rangle : x_\alpha \in C_\beta^s \} \). Since \( Y = \bigcup_{f \in \omega^<\omega} \bigcap_n \omega^<\omega B_{f \upharpoonright n} \) it is enough to check that each \( B_s \in B(R) \). Fix \( s \in \omega^<\omega \) and let \( \{ D_n : n < \omega \} \) be an open basis for \( X \). For each \( \beta \) define \( y_\beta(n) = 1 \) if \( D_n \cap C_\beta^s = \emptyset \). It follows that \( \alpha \in C_\beta^s \) if \( \forall n \) (if \( y_\beta(n) = 1 \) then \( \alpha \in D_n \)). Letting \( E_n = (D_n \times X) \cup (D_n \times \{ \beta : y_\beta(n) = 0 \}) \) we have that \( B_s = \bigcap_n \omega^<\omega E_n \).

**Lemma 8.** Suppose \( F : X \to Y \) is 1-1 and \( \forall U \) open in \( Y \) \( F^{-1}(U) \) is Borel in \( X \). If every subset of \( Y \) is analytic in \( Y \) then every subset of \( X \) is analytic in \( X \).

**Proof.** Given \( A \subseteq X \) let \( B = F''A \). Then there are Borel subsets of \( Y, B_s \) for \( s \in \omega^<\omega \) such that \( B = \bigcup_{f \in \omega^<\omega} \bigcap_n \omega^<\omega B_{f \upharpoonright n} \). Let \( A_s = F^{-1}(B_s) \), then \( A_s \) is Borel in \( X \) and \( A = \bigcup_{f \in \omega^<\omega} \bigcap_n \omega^<\omega A_{f \upharpoonright n} \). \( \square \)

We now prove Theorem 2. Let \( M \), the ground model of ZFC in \( \S 2 \), be a model of \( MA + 2^\omega = \omega_1 \). We first show that for \( G_{\omega_2} \) \( \text{Q}_{\omega_2} \)-generic over \( M, M[G_{\omega_2}] \) models that \( S(R) = P(2^\omega \times 2^\omega) \). Working in \( M \) for any \( Z, W \subseteq 2^\omega \) with \( |Z| = |W| = \omega_1 \), if \( F : Z \to W \) is any \( 1-1 \) map then by Silver's lemma (see [6]) for every \( U \) open in \( W, F^{-1}(U) \) is Borel in \( Z \). \( F \) still has this property in any extension of \( M \) since \( W \) is second countable and \( M \) contains an open basis for \( W \). Working in \( M \) there exists \( X \subseteq 2^\omega \) such that \( |X| = \omega_1 \) and \( \forall Y \subseteq X \) if \( |Y| \leq \omega_1 \) then \( Y \) is Borel in \( X \) (a generalized Luzin set is such an example, see [9]). We claim that in \( M[G_{\omega_2}] \) every subset of \( X \) of size \( \leq \omega_1 \) is analytic in \( X \) and thus by Lemma 7, \( S(R) = P(2^\omega \times 2^\omega) \). Working in \( M[G_{\omega_2}] \) for any \( Z \subseteq X \) if \( |Z| \leq \omega_1 \), then \( \exists Y \in MZ \subseteq Y \) and \( |Y| \leq \omega_1 \). Letting \( F : Y \to X^* \) be any \( 1-1 \) map in \( M \) we have by Lemma 8 that every subset of \( Y \) is analytic in \( Y \), and since \( Y \) is Borel in \( X, Z \) is analytic in \( X \).

We next want to show that in \( M[G_{\omega_2}], P(2^\omega \times 2^\omega) \neq B(R) \). It is enough to show that in \( M[G_{\omega_2}] \) there does not exist a countable \( H \subseteq P(X^*) \) such that \( B(H) = P(X^*) \). To see that this suffices let \( \{ X_\alpha : \alpha < \omega_1 \} \) be a generic Souslin set over \( M \) and let \( Y = \{ \langle x, \alpha \rangle : x \in X_\alpha \} \subseteq X^* \times 2^\omega \). If \( Y \) is in the \( \sigma \)-algebra generated by \( \{ A_n \times B_n : n < \omega \} \) then \( B(\{ A_n \times B_n : n < \omega \}) \neq P(X^*) \). Just show by induction that \( \forall K \in B(\{ A_n \times B_n : n < \omega \}) \forall \beta < \omega, \{ x \in X^* : (x, \beta) \in K \} \in B(\{ A_n : n < \omega \}) \).

By the technique of \( \S 1 \) and \( \S 2 \) we note that in \( M \) there is no countable \( H \subseteq P(X^*) \) such that the generic Souslin set created at the first step is in \( B(H) \). Note that for \( Z = \emptyset \) and \( G P(Z) \)-generic over \( M \) the set \( A = \{ x_\alpha \in X^* : G_\alpha(\langle 0 \rangle) = 1 \} \) is also a generic Souslin set over \( M \). This is because the requirement that \( G_\alpha(\emptyset) = 0 \) puts no constraint on the value of \( G_\alpha(\langle 0 \rangle) \). \( \square \)
4. Remarks. (1) In the model used for Theorem 1 one can show that there does not exist any $H \subseteq P(2^\omega)$, $|H| < |2^\omega|$, such that every analytic subset of $2^\omega$ is in $B(H)$. Note also that $\omega_1$ can be replaced by any $\kappa > \omega_1$ of uncountable cofinality. Also in this model it is true that the universal $\Sigma^1_1$ subset of $2^\omega \times 2^\omega$ is not in the $\sigma$-algebra generated by the abstract rectangles.

(2) It is not hard to modify the technique of §2 to get it consistent with ZFC that $3lg\omega_1=\omega_2$ (or even $|X| = \aleph_\omega$) such that every subset of $X$ is analytic in $X$ but not every subset of $X$ is Borel in $X$.

(3) $X^*$ in §2 has Baire order $\omega_1$ in $M[G_{\omega_1}]$.

(4) In [5] Kunen showed that if one adds $\omega_1$ Cohen reals to a model of CH then $\{<\alpha, \beta>: \alpha < \beta < \omega_1\}$ is not in the $\sigma$-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_1\}$. In the same model (actually CH is not necessary in ground model) there is a subset of $\omega_1 \times \omega_2$ not in the $\sigma$-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$. To prove this it is enough to find $F \subseteq P(\omega_1)|F| = \omega_2$ such that there does not exist $H \subseteq P(\omega_1)$ countable with $F \subseteq B(H)$. Let $P = \{p \mid p: F \to 2, \text{for some } F \in [\omega_1]^{<\omega}\}$ and suppose $G$ is $P$-generic over $M$. Let

$$X = \{\alpha < \omega_1 \mid G(\alpha) = 1\}$$

and note that for any $H \subseteq P(\omega_1)$ countable and in $M, M[G] \models "X \in B(H)"$. This is because for any $Y \in B(H)$ $\exists t \in 2^\omega Y \in M[t]$.

(5) In [12] Rothberger showed that $2^\omega = \omega_3 + 2^\omega = \aleph_\omega$ implies that not every subset of $\omega_1 \times \omega_2$ is in the $\sigma$-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$. To see this let $G_\alpha$ for $\alpha < \aleph_\omega$ list all countable subsets of $P(\omega_1)$. Since $|B(G_\alpha)| \leq 2^\omega = \omega_1$ we can pick $K_\alpha \in P(\omega_1)$ for $\alpha < \omega_1$ such that $K_\alpha \in \bigcup_{\beta < \omega_1} B(G_\beta)$. It follows as in (4) that $\{(\beta, \alpha): \beta \in K_\alpha\}$ is not in the $\sigma$-algebra generated by $\{A \times B: A \subseteq \omega_1, B \subseteq \omega_2\}$.

References


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