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SOME PROBLEMS IN SET THEORY AND MODEL THEORY.

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1978
Some Problems in Set Theory and Model Theory

By

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INTRODUCTION AND ACKNOWLEDGEMENTS

It is often the case that it is easier to prove something is consistent with ZFC than it is to prove it. This is especially true when it is independent of ZFC.

The focus of this thesis is on subsets of the set of real numbers and some of their properties. By a real number we could mean an element of $2^\omega$ the Cantor space or $\omega^\omega$ the Baire space (topologized as usual by basic open sets of the form $[s] = \{ f \in \omega^\omega : s \subseteq f \}$ where $s \in \omega^\omega = \bigcup_{n<\omega} n^\omega$). Also a countable structure is a real number. For example, identify a structure $<\omega, R>$ where $R \subseteq \omega^2$ a binary relation with an element of $2^{\omega \times \omega}$. Even an ultrafilter on $\omega$ is just a set of real numbers with some peculiar property.

Some important notation:

- $|A|$ usually denotes the cardinality of the set $A$ except sometimes it's used to denote the universe of a model, the rank of an element in a well-founded tree, or any two of the above.

- $\Sigma^0_0 = \Pi^0_0 = \Delta^0_1$ = clopen sets of reals.
- $\Sigma^0_\alpha = \text{countable unions of } \bigcup_{\beta<\alpha} \Pi^0_\beta$.
- $\Pi^0_\alpha = \text{countable intersections of } \bigcup_{\beta<\alpha} \Sigma^0_\beta$.
- $= \text{complements of } \Sigma^0_\alpha$'s.
- $\Sigma^0_\delta = G_{\delta\sigma\delta\sigma}$.
- $P(X)$ denotes the set of all subsets of $X$. 

Each of the four Parts begins with an introduction and ends with a bibliography.

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I would like to dedicate this thesis to my wife, Man-Li.
Introduction.

For any separable metric space \( X \) and \( \alpha \) with \( 1 \leq \alpha \leq \omega_1 \) define the Borel classes \( \Sigma_\alpha^0 \) and \( \Pi_\alpha^0 \). Let \( \Sigma_1^0 \) be the class of open sets and for \( \alpha > 1 \) \( \Sigma_\alpha^0 \) is the class of countable unions of elements of 
\[ U\{\Pi_\beta^0 : \beta < \alpha\} \text{ where } \Pi_\beta^0 = \{X - A : A \in \Sigma_\beta^0\}. \]
Hence 
\[ \Sigma_1^0 = \text{open} = G, \quad \Pi_1^0 = \text{closed} = F, \quad \Sigma_2^0 = F_\sigma, \quad \Pi_2^0 = G_\delta, \text{ etc.} \]
Note that \( \Sigma_{\omega_1}^0 = \Pi_{\omega_1}^0 = \text{set of all Borel in } X \text{ subsets of } X. \)
The Baire order of \( X \) (ord(\( X \))) is the least \( \alpha \leq \omega_1 \) such that every Borel in \( X \) subset of \( X \) is \( \Sigma_\alpha^0 \) in \( X \). Since the Borel subsets of \( X \) are closed under complementation we could equally well have defined ord(\( X \)) in terms of \( \Pi_\alpha^0 \) in \( X \) or \( \Delta_\alpha^0 = \Pi_\alpha^0 \cap \Sigma_\alpha^0 \) in \( X \). Note also that for \( X \subseteq \mathbb{R} \) (the real numbers) ord(\( X \)) is the least \( \alpha \) such that for every Borel set \( A \) in \( \mathbb{R} \) there is a \( \Sigma_\alpha^0 \) in \( \mathbb{R} \) set \( B \) such that \( A \cap X = B \cap X. \) Also note that ord(\( X \)) = 1 iff \( X \) is discrete, ord(\( Q \)) = 2 where \( Q \) is the space of rationals, and in general for \( X \) a countable metric space ord(\( X \)) \leq 2 since every subset of \( X \) is \( \Sigma_2^0(F_\sigma) \) in \( X. \)

It is a classical theorem of Lebesgue (see [11]) that for any uncountable Polish (separable and completely
metrizable) space \( \text{ord}(X) = \omega_1 \). The same is true for any uncountable analytic \( (\Sigma^1_1) \) space \( X \) since \( X \) has a perfect subspace (see [11]) and Borel hierarchies relativize.

The Baire order problem of Mazurkiewicz (see [19]) is: for what ordinals \( \alpha \) does there exist \( X \subset \mathbb{R} \) such that \( \text{ord}(X) = \alpha \). Banach conjectured (see [29]) that for any uncountable \( X \subset \mathbb{R} \) the Baire order of \( X \) is \( \omega_1 \). In §3 we review the classically known results of Sierpinski, Szpilrajn, and Poprougenko. We show that it is consistent with ZFC that for each \( \alpha \leq \omega_1 \) there is an \( X \subset \mathbb{R} \) with \( \text{ord}(X) = \alpha \). In fact, we prove a theorem of Kunen's that CH implies this. We also show that Banach's conjecture is consistent with ZFC.

Given a set \( X \) and \( R \) a family of subsets of \( X \) \((R \subset P(X)) \) define for every \( \alpha \leq \omega_1 \) \( R_\alpha \subset P(X) \) as follows. Let \( R_0 = R \) and for each \( \alpha > 0 \) if \( \alpha \) is even (odd) let \( R_\alpha \) be the family of countable intersections (unions) of elements of \( \bigcup \{ R_\beta : \beta < \alpha \} \). Generalizing Mazurkiewicz's question Kolmogorov (see [8]) asked: for what ordinals \( \alpha \) does there exist \( X \) and \( R \subset P(X) \) such that \( \alpha \) is the least such that \( R_\alpha = R_{\omega_1} \). Kolmogorov's question can be generalized by replacing \( P(X) \) by an arbitrary \( \sigma \)-algebra (a countably complete boolean algebra). In §2 we prove that for any \( \alpha \leq \omega_1 \) there is a complete boolean algebra with the countable chain condition which is countably generated.
in exactly $\alpha$ steps. This answers a question of Tarski who had noticed that the boolean algebras $\text{Borel}(2^\omega)$ modulo the ideal of meager sets and $\text{Borel}(2^\omega)$ modulo the ideal of measure zero sets are countably generated in exactly one and two steps respectively (see [4]). Theorem 12 which is due to Kunen shows that the same answer to Kolmogorov's problem (every $\alpha \leq \omega_1$) follows from the solution of Tarski's problem.

Let $R = \{ A \times B : A, B \subseteq 2^\omega \}$. In §4 we show that for any $\alpha$, $2 \leq \alpha < \omega_1$, it is consistent with ZFC that $\alpha$ is the least ordinal such that $R_\alpha$ is the set of all subsets of $2^\omega \times 2^\omega$. This answers a question of Mauldin [1].

For $\alpha \leq \omega_1$ a set $X \subseteq 2^\omega$ is a $Q_\alpha$ set iff every subset of $X$ is Borel in $X$ and $\text{ord}(X) = \alpha$. It is shown that it is consistent with ZFC that for every $\alpha < \omega_1$ there is a $Q_\alpha$ set. In §4 we also show that there are no $Q_{\omega_1}$ sets. However, we do show that it is consistent with ZFC that there is an $X \subseteq 2^\omega$ with $\text{ord}(X) = \omega_1$ and every $X$-projective set is Borel in $X$. This answers a question of Ulam [31], p. 10.

Also in §4 we show that it is relatively consistent with ZFC that the universal $\Sigma^1_1$ set is not in $R_{\omega_1}$ confirming a conjecture of Mansfield [13] who had shown that the universal $\Sigma^1_1$ set is never in the $\sigma$-algebra generated by the rectangles with $\Sigma^1_1$ sides.
Given \( R \subseteq P(X) \) let \( K(R) \) (the Kolmogorov number of \( R \)) be the least \( \alpha \) such that \( R_\alpha = R_{\omega_1} \). It is an exercise to show that for \( \alpha = 0,1, \) or 2 there is an \( R \subseteq P(\{0,1\}) \) with \( K(R) = \alpha \).

**Proposition 1.** Given \( R \subseteq P(X) \) then (a) if \( R \) is finite or \( X \) is countable then \( K(R) \leq 2 \), and (b) there exists \( S \subseteq P(Y) \) such that cardinality of \( S \) and \( Y \) is \( \leq 2^{\omega_0} \) and \( K(R) = K(S) \).

**Proof.**

(a) Note \( \bigcup_{\alpha<\alpha_0} \bigcap_{\beta<\beta_0} \bigcup_{\gamma<\gamma_0} A_{\alpha,\beta,\gamma} = f(\bigcap_{\alpha<\alpha_0} \bigcup_{\gamma<\gamma_0} A_{\alpha},\gamma) \), \( f:\alpha_0 \to \beta_0 \) can always be taken to be a countable intersection.

(b) Let \( V_\alpha \) be the sets of rank less than \( \alpha \). Choose \( \alpha \) a limit ordinal of uncountable cofinality so that \( R, X \in V_\alpha \). Let \((M,\varepsilon)\) be an elementary substructure of \((V_\alpha,\varepsilon)\) containing \( R \) and \( X \) such that \( M^\omega \subseteq M \) and \( |M| \leq 2^{\omega_0} \). Now let \( Y = X \cap M \) and \( S = \{A \cap Y: A \in R \cap M\} \).

Mazurkiewicz's problem is equivalent to Kolmogorov's problem for \( R \) a countable field of sets (that is closed under finite intersection and complementation).
Proposition 2. (Sierpinski [23] also in [30]) Given $R \subseteq P(X)$ a countable field of sets there exists $Y \subseteq 2^\omega$ such that $K(R) = \text{ord}(Y)$. (That is we may reduce to considering subsets $Y$ of $2^\omega$ and relativizing the usual Borel hierarchy on $2^\omega$ to $Y$.)

Proof. Let $R = \{A_n : n \in \omega\}$ and define $F: X \to 2^\omega$ by $F(x)(n) = 1$ iff $x \in A_n$. Put $Y = F''X$.

Define $K = \{\beta : 2 \leq \beta < \omega_1$ and there is $X \subseteq \omega^\omega$ uncountable with $\text{ord}(X) = \beta\}$. What can $K$ be?

Proposition 3. $K$ is a closed subset of $\omega_1$.

Proof. Given $A \subseteq \omega^\omega$ and $n \in \omega$ define $nA = \{x \in \omega^\omega : x(0) = n$ and $\exists y \in A \forall n(x(n + 1) = y(n))\}$. If $X = \bigcup_{n \in \omega} nX_n$, then it is readily seen that $\text{ord}(X) = \sup_{n \in \omega} \text{ord}(X_n)$. 

Note that $K$ is the same set of ordinals if we replace $\omega^\omega$ by $R$ the real numbers or $2^\omega$. This is true for $R$ because if $X \subseteq R$ and $R - X$ is not dense then $X$ contains a nonempty interval, hence $\text{ord}(X) = \omega_1$; but $R - X$ dense means we may as well assume $X \subseteq \text{irrationals} \equiv \omega^\omega$.

In the definition of $K(R) = \omega$ for $R \subseteq P(X)$ we ignored the possibility that the hierarchy on $R$ might
have exactly $\omega$ levels, i.e. $R_{\omega_1} = \bigcup \{ R_n : n < \omega \}$ but for all $n < \omega$, $R_n \neq R_{\omega_1}$. In fact a Borel hierarchy of length less than $\omega_1$ must have a top level.

**Proposition 4.** If $R \subseteq P(X)$ is a field of sets, $\lambda$ is a countable limit ordinal, and $R_{\omega_1} = \bigcup \{ R_\alpha : \alpha < \lambda \}$ then there is $\alpha < \lambda$ such that $R_\alpha = R_{\omega_1}$.

**Proof.**

Using the proof of Proposition 2 we can assume $X \subseteq 2^K$ for some $\kappa$ and $R = \{ [s] \cap X : S : D + 2 \text{ where } D \subseteq \kappa \text{ is finite} \}$ where $[s] = \{ f \in 2^K : f \text{ extends } s \}$. For each $A$ in $R_{\omega_1}$ there is $T \subseteq \kappa$ countable such that for any $f$ and $g$ in $X$ if $f|T = g|T$, then $f \in A$ iff $g \in A$. In this case we say $T$ supports $A$. Choose $T \subseteq \kappa$ countable so that for any $D \subseteq T$ finite and $s : D + 2$ if $\text{ord}(X \cap [s]) = \lambda$ then for any $\alpha < \lambda$ there is an $A \subseteq [s]$ in $R_{\alpha+1} = R_\alpha$ such that $T$ supports $A$. By taking an autohomeomorphism of $2^K$ we may assume $T = \omega$.

Define $L$ to be $\{ s \in 2^{<\omega} : \text{ord}(X \cap [s]) = \lambda \}$.

**Claim.** For any $s$ in $L$ there are $t$ and $\hat{t}$ in $L$ incompatible extensions of $s$.

**Proof.**

Without loss of generality assume $s = \emptyset$ and there is $f \in 2^\omega$ such that for every $s \in L$ $s \subseteq f$. For each $n < \omega$
define \( t_n \) in \( 2^{n+1} \) by \( t_n(m) = f(m) \) for \( m < n \) and \( t_n(n) = 1 - f(n) \). Then \( [f] \cup \bigcup \{[t_n]: n < \omega\} \) is a disjoint union covering \( 2^{\lambda} \). If there is a \( \beta_0 < \lambda \) such that for all \( n < \omega \) \( \text{ord}([t_n] \cap X) < \beta_0 \), then for all \( A \) in \( R_{\omega_1} \) supported by \( \omega \) \( A \) is in \( R_{\beta_0+1} \). This is because \( A \cap [f] = \emptyset \) or \( X \cap [f] \subseteq A \). But this contradicts the choice of \( \omega \).

On the other hand, if there is no such bound \( \beta_0 \), choose \( Z_n \subseteq [t_n] \) with \( Z_n \in R_{\omega_1} \) so that for every \( \beta < \lambda \) there is \( n < \omega \) with \( Z_n \not\in R_\beta \). But then \( \bigcup \{Z_n: n < \omega\} \) is not in \( \bigcup \{R_\beta: \beta < \lambda\} \). This proves the claim and this last argument also proves the proposition from the claim.

Remark. If \( R \subseteq \mathcal{P}(X) \) and \( R_{\omega_1} = \bigcup \{R_n: n < \omega\} \) and there is \( n_0 < \omega \) such that \( \{X - A: A \in R\} \subseteq R_{n_0} \), then there is \( n_1 < \omega \) such that \( R_{n_1} = R_{\omega_1} \). Willard [32] shows that for any \( \alpha < \omega_1 \) there are \( R \) and \( X \) with \( R \subseteq \mathcal{P}(X) \) such that \( \alpha \) is the least ordinal such that \( \{X - A: A \in R\} \subseteq R_\alpha \).
§1. Some basic definitions and lemmas

For $T \subseteq \omega^<\omega$ $T$ is a well founded tree iff $T$ is a tree (if $t \subseteq s \in T$ then $t \in T$) and is well founded (for any $f \in \omega^\omega$ there is an $n < \omega$ such that $\forall n \notin T$).

For $s \in T$ define $|s|^T$ (the rank of $s$ in $T$) by $|s|^T = \sup\{|t|^T + 1: s \subseteq t \in T\}$. Often we drop $T$ and let $|s| = |s|^T$. $T$ is normal of rank $\alpha$ means that:

(a) $T$ is a well founded tree;
(b) $|\emptyset| = \alpha$ ($\emptyset$ is the empty sequence);
(c) ($s \in T$ and $|s| > 0$) $\rightarrow$ ($\forall i < \omega (s^\omega i \in T)$);
(d) ($s \in T$ and $|s| = \beta + 1$) $\rightarrow$ ($\forall i < \omega (|s^\omega i| = \beta)$);
(e) ($s \in T$ and $|s| = \lambda$ where $\lambda$ is a limit ordinal) $\rightarrow$

$\forall \beta < \lambda \{i: |s^\omega i| < \beta\} \text{ is finite and }\forall i < \omega |s^\omega i| \geq 2$.

Note that for any $n < \omega$ the tree $\omega^<n$ is normal of rank $n$. If $\alpha_n$ for $n < \omega$ are strictly increasing to $\alpha$ (or $\alpha_n = \beta$ where $\alpha = \beta + 1$) and for each $n < \omega$ $T_n$ is normal of rank $\alpha_n$, then $T = \bigcup n^{<\omega} s: n < \omega$ and $s \in T_n$ is normal of rank $\alpha$. We often use $T_\alpha$ to denote some fixed normal tree of rank $\alpha$.

For any $\alpha < \omega_1$ and $Y \subseteq X \subseteq \omega^\omega$ define the partial order $\mathcal{P}_\alpha(Y,X)$ (the order is given by inclusion). Fix some $T$
normal of rank \( \alpha. \) \( p \in P_\alpha(Y, X) \) iff \( p \subseteq (T - \{\phi\}) \times (X \cup \omega^\omega) \) and (1) through (5) hold.

(1) \( p \) is finite.

(2) \(|s| = 0\) implies that if \((s, x) \in p\) then \(x \in \omega^\omega\) and if \((s, y) \in p\) then \(x = y\). (So if \(T^* = \{s \in T: |s| = 0\}\) then \(p \upharpoonright (T^* \times (X \cup \omega^\omega))\) is a function from a finite subset of \(T^*\) into \(\omega^\omega\).)

(3) If \(|s| > 0\) and \((s, x) \in p\) then \(x \in X\).

(4) If \(s\) and \(s^i \in T\) and \(x \in X\) then not both \((s, x)\) and \((s^i, x)\) are in \(p\), or if \(|s^i| = 0\), not there exists \(k \in \omega\) such that both \((s, x)\) and \((s^i, x \upharpoonright k)\) are in \(p\).

(5) If \(s\) of length one and \((s, x) \in p\) then \(x\) is not in \(Y\).

Let \(G\) be \(P_\alpha(Y, X)\) generic. Working in the extension define for each \(s \in T, G_s \subseteq \omega^\omega\). For \(|s| = 0\), let \(G_s = \{x \in \omega^\omega: \exists t \in \omega^\omega \ t \leq x \text{ and } \{(s, t)\} \in G\}\).

For \(|s| > 0\), let \(G_s = \{\omega^\omega - G_s \cap i: i < \omega\}\).

Note that for each \(s \in T\)
\(G_s \subseteq \prod_{|s|}^0\).

**Lemma 5.** For each \(x\) in \(X\) and \(s\) in \(T - \{\phi\}\) with \(|s| > 0\[x \in G_s\] iff \{(s, x)\} \in G\].
Proof.

Case 1. \(|s| = 1\). (This is the argument from almost-disjoint-sets forcing.)

If \(x \in G_s\) then \(x \not\in G_s^{\alpha_i}\) for all \(i \in \omega\). Hence for all \(k, i\) in \(\omega\), \((s^{\alpha_i}, x^k) \not\in G\). Let \(D = \{p: (s, x) \in p\) or there exist \(k, i\) such that \((s^{\alpha_i}, x^k) \in p\}\). \(D\) is dense since if \((s, x) \not\in p\) if we let \(\{x_1, x_2, \ldots, x_n\} \subseteq X\) be all the elements of \(\omega\) mentioned in \(p\) other than \(x\), we can choose \(k\) sufficiently large so that \(x^k \not\in x^k_i\) for all \(i \leq n\). Also we can choose \(j\) sufficiently large so that \((s^\alpha j)\) is not mentioned in \(p\) and then \(p \cup \{(s^{\alpha_j}, x^k)\} \in (\mathcal{P}_\alpha(Y, X) \land D)\).

Since \(G \land D\) is non-empty and \(x \not\in G_s^{\alpha_i}\) all \(i\); we conclude that \((s, x) \in G\).

If \(x \not\in G_s\) then \(x \in G_s^{\alpha_i}\) for some \(i\). Hence there exist \(k\) such that \((s^{\alpha_i}, x^k) \in G\) so \((s, x) \not\in G\) by clause \((4)\).

Case 2. \(|s| > 1\).

If \(x \in G_s\) then \(x \not\in G_s^{\alpha_i}\) for all \(i\), and hence by induction \((s^{\alpha_i}, x) \not\in G\) for all \(i\). Let \(D = \{p: (s, x) \in p\) or there exist \(i\) such that \((s^{\alpha_i}, x) \in p\}\). \(D\) is dense hence \((s, x) \in G\).

If \(x \not\in G_s\) then \((s^{\alpha_i}, x) \in G\) for some \(i\) (by induction). Hence \((s, x) \not\in G\) by clause \((4)\).
Corollary 6. Gφ \land X = Y (\omega_2)

Proof.
If x \in Y then for every n,((n),x) \notin G (by clause 5).
Hence by Lemma 5 for every n,x \notin G(n) and so x \in Gφ.
If x \notin Y then \{p: \text{there exists } n \text{ such that } ((n),x) \in p\}
is dense hence there exists n such that x \in G(n)
(by Lemma 5) so x \notin Gφ.

Remarks:
(1) \mathcal{P}_0(X,Y) is trivial (the empty set).
(2) \mathcal{P}_1(X,Y) has nothing to do with X and Y and is
isomorphic as a partial order to the usual Cohen partial
order for adding a map from \omega to \omega.
(3) \mathcal{P}_2(X,Y) is another way of viewing Solovay's "almost-disjoint-sets forcing" (see [6]).

Lemma 7. \mathcal{P}_\alpha(X,Y) has the countable chain condition.

Proof.
Suppose by way of contradiction that there exist F included in \mathcal{P}_\alpha(X,Y) of cardinality \aleph_1 of pairwise incompatible conditions. Since there are only countably many
finite subsets of T, we may assume there exist H \subseteq T - \{\phi\}
finite so that every p \in F is included in
H \times (X \cup \omega^\omega). We may also assume that for every p \in F
and q \in F and s \in H with |s| = 0 and t \in \omega^\omega that
[(s, t) ∈ p iff (s, t) ∈ q]. Now let \((x_β : β < \aleph_1)\) be all the elements of \(X\) occurring in members of \(F\). For each \(p\) in \(F\) let \(p^*: G_p \rightarrow P(H)\) be defined by
\[
G_p = \{β: \text{there exists } s \ (s, x_β) \in p\} \quad \text{and for } β \in G_p \quad p^*(β) = \{s: (s, x_β) \in p\}.
\]
\([p^*: p \in F]\) is a family of incompatible conditions in the partial order \(Q\), where
\(Q = \{p: \text{domain of } p \text{ is a finite subset of } \aleph_1 \text{ and range of } p \text{ is } P(H)\}\), ordered by inclusion. Since it is well known that \(Q\) has the countable chain condition we have a contradiction. 

Remarks:
1) If \(P = P_\alpha(Y, X)\) for any \(α, X,\) and \(Y\) then \(P\) is absolutely c.c.c. That is to say if \(P \in M \models "\text{ZFC}"\) then \(M \models "P \text{ has c.c.c.}"\). It follows that the direct sum of any combination of the \(P_\alpha\)'s has the c.c.c. (direct sum of \(Q_α: α < κ\)) is \(\bigoplus_{α<κ} Q_α = \{p: p: κ → \bigcup_{α<κ} Q_α, \forall α < κ \quad p(α) \in Q_α, \text{ and } p(α) = 0 \text{ for all but finitely many } α\}. p \geq q \iff \forall α(p(α) ≥ q(α)).\n
2) We assume the fact that iterated c.c.c. forcing is c.c.c. (Solovay-Tennenbaum [26]) and occasionally use notation and facts from [26].

I would like to prove next an heuristic proposition. Define \(P\) a partial order: \(p \in P\) iff \(p\) is a finite consistent set of sentences of the form "\([s] \subseteq G_n\),"\(x \notin G_n\),
or \( x \in \bigcap_{n \in \omega} G_n \) (where \( s \in \omega^\omega \) and \( x \in \omega \)). Order \( \mathbb{P} \) by inclusion. Any \( G \mathbb{P} \)-generic determines a \( \Pi^0_2 \) set \( \bigcap_{n \in \omega} G_n \).

**Proposition.** If \( G \) is \( \mathbb{P} \)-generic over \( M \) (transitive countable model of ZFC) then \( M[G] \models \forall F \in F \forall (F \in M \neq \bigcap_{n \in \omega} G_n \land M) \).

**Proof.**

Suppose not and let \( C_n \) be closed for \( n \in \omega \) and \( p \in G \) be such that \( p \models \forall \bigcup_{n \in \omega} C_n \land M = \bigcap_{n \in \omega} G_n \land M \). It is easily seen that \( \mathbb{P} \) has c.c.c. (see the proof of Lemma 7). Thus working in \( M \) we can find \( Q \subseteq P \) countable such that for any \( \hat{G} \mathbb{P} \)-generic, \( n \in \omega \), and \( s \in \omega^\omega \), if \( M[\hat{G}] = \left[ [s] \land C_n = \emptyset \right] \) then \( \exists q \in Q \land \hat{G} \) such that \( q \models [s] \land C_n = \emptyset \). Since \( Q \) is countable, we can find \( z \in \omega \land M \) not mentioned in \( p \) or any condition in \( Q \).

Since \( p \lor \{ z \in \bigcap_{n \in \omega} G_n \} \models [z \in \bigcup_{n \in \omega} C_n] \), we can find \( \tilde{n} \in \omega \) and \( \tilde{p} \geq p \) and not mentioning \( z \) so that \( \tilde{p} \lor \{ z \not\in \bigcap_{n} G_n \} \models [z \not\in C_{\tilde{n}}] \), because the only other way to mention \( z \) is \( [z \not\in G_n] \). By taking \( \tilde{m} \) large enough \( \tilde{p} \lor \{ z \not\in G_{\tilde{m}} \} \) will be consistent, and since it extends \( p \) it forces \( [z \not\in C_{\tilde{n}}] \). Let \( G \) be \( \mathbb{P} \)-generic with \( \tilde{p} \lor \{ z \not\in G_{\tilde{m}} \} \) in \( G \). Let \( k \in \omega \) and \( q \in G \land Q \) be so that \( q \models [z \land k] \land C_{\tilde{n}} = \emptyset \). But \( \tilde{p} \lor q \lor \{ z \in \bigcap_{n \in \omega} G_n \} \) is consistent because \( q \in Q \) and so doesn't mention \( z \). This
is a contradiction since \( q \models \neg "z \notin C_n" \) and
\[
\hat{p} \cup \{ z \in \bigcup_{n \in \omega} C_n \} \models \neg "z \in C_n".
\]

Define for \( F \subseteq \omega^\omega \) and \( p \in \mathbb{P} = \mathbb{P}_\alpha(Y, X) \),
\[
|p|(F) = \max(|s| : \text{there is } x \notin F \text{ with } (s, x) \in p).
\]
This is called the rank of \( p \) over \( F \).

**Lemma 8.** For all \( \beta \geq 1 \) and \( p \in \mathbb{P} \) there is \( \hat{p} \in \mathbb{P} \)
compatible with \( p \) and \( |\hat{p}|(F) < \beta + 1 \) so that for any
\( q \in \mathbb{P} \) with \( |q|(F) < \beta \), if \( \hat{p} \) and \( q \) are compatible then
\( p \) and \( q \) are compatible.

**Proof.** First find an extension \( p_0 \supseteq p \) so that for all
\( (s, x) \in p \) and \( i < \omega \) if \( |s| = \lambda \) is a limit ordinal and
\( |s^i| < \beta + 1 < \lambda \) (there are only finitely many such \( s^i \)),
then there is a \( j < \omega \) such that \( (s^i, x) \in p_0 \). Now let
\( \hat{p} = \{ (s, x) \in p_0 : |s| < \beta + 1 \text{ or } x \in F \} \). We check that \( \hat{p} \)
has the requisite property. Suppose \( p \) and \( q \) are in­
compatible, \( \hat{p} \) and \( q \) are compatible, and \( |q|(F) < \beta \).
Since \( \beta \geq 1 \) for all \( (s, x) \in p \) if \( |s| < 1 \) then
\( (s, x) \in \hat{p} \), hence since \( \hat{p} \) and \( q \) are compatible there are
\( s, t \in \omega^\omega \), \( i < \omega \), and \( x \in \omega^\omega \) such that
\( (s, x) \in p \),
\( (t, x) \in q \), and \( s = t^i \text{ or } t = s^i \).

**Case 1.** If \( x \in F \) or \( |s| < \beta + 1 \) then \( (s, x) \in \hat{p} \) and
so \( \hat{p} \) and \( q \) are incompatible.

**Case 2.** If \( x \notin F \) and \( |s| \geq \beta + 1 \) then by definition of
|q| < \beta, \ |t| < \beta. So \ t = s^i. If \ |s| = \gamma + 1 \ for 

some \ \gamma \ then \ |t| = \gamma \geq \beta, \ contradiction. If \ |s| = \lambda \ is 
an infinite limit ordinal then by the construction of \ p_0 
there is \ j < \omega \ with \ (t^j, x) \in \ p_0 \ and \ hence \ (t^j, x) \in \hat{p} 
and so \ q \ and \ \hat{p} \ are incompatible.

§ 2 Boolean Algebras

For \ B \ a complete boolean algebra, \ C \ included in 
B, and \ \alpha \geq 1 \ define \ \Sigma_\alpha(C), \ \Pi_\alpha(C): 

\Sigma_1(C) = \{S: S \subseteq C\}, 
\Sigma_\alpha(C) = \{S: S \subseteq \bigcup_{\beta<\alpha} \Pi_\beta(C)\} \ for \ \alpha > 1 \, \text{and} 
\Pi_\alpha(C) = \{-a: a \in \Sigma_\alpha(C)\} 

Define \ K(B) \ to be the least ordinal \ \alpha \ such that there 
exists a countable \ C \ included in \ B \ with \ \Sigma_\alpha(C) = B. 

Theorem 9. For each \ \alpha \leq \omega_1 \ there exists a complete 
boolean algebra \ B \ with countable chain condition and 
\ K(B) = \alpha. 

Proof. 
For \ \alpha = 0 \ take \ B \ to be any finite boolean algebra. 
For \ \alpha = 1 \ take \ B \ to be \ (P(\omega), \wedge, \vee) \ (or more ap-
propriately the regular open subsets of \ \omega^\omega \ since this 
corresponds to Cohen real forcing).
For $\alpha$, $2 \leq \alpha < \omega_1$, $\mathcal{B}$ will be the complete boolean algebra associated with $\prod_{\alpha}^0$-forcing. Let $\mathcal{P} = \mathcal{P}_\alpha(\emptyset, X)$. Given a partial order $\mathcal{P}$ there is a canonical way of constructing a complete boolean algebra $\mathcal{B}$ in which $\mathcal{P}$ is densely embedded (see [5]). Let $[p]$ denote the image of $p \in \mathcal{P}$ under this embedding $j$ then if $p \geq q$ then $[p] \leq [q]$, and for every $a \in \mathcal{B}$ if $a \neq 0$ then there is a $p \in \mathcal{P}$ such that $[p] \leq a$.

**Lemma 10.** Suppose $F \subseteq X$ and $C = \{[p]: p \in \mathcal{P}$ and $|p|(F) = 0\}$. For any $\beta \geq 1$, $p \in \mathcal{P}$, and $a$ in $\Sigma_\beta(C)$, if $[p] \leq a$ then there is $q \in \mathcal{P}$ such that $|q|(F) < \beta$, $q$ and $p$ are compatible, and $[q] \leq a$.

**Proof.**

The proof is by induction on $\beta$.

**Case 1.** $\beta = 1$. Suppose $a = \Sigma\{[q]: q \in \Gamma\}$ for some $\Gamma \subseteq C$. If $[p] \leq a$ then for $q \in \Gamma$, $p$ and $q$ are compatible.

**Case 2.** $\beta$ a limit ordinal. Suppose $a = \Sigma\{b: b \in \Gamma\}$ for some $\Gamma \subseteq U(\Sigma_\alpha(C): \alpha < \beta)$. Then there is $\hat{p} \geq p$ and $b \in \Gamma \cap \Sigma_\alpha(C)$ for some $\alpha < \beta$ so that $[\hat{p}] \leq b$. Now apply the inductive hypothesis to $\hat{p}$.

**Case 3.** $\beta + 1$. Suppose $[p] \leq \Sigma\{b: b \in \Gamma\}$ for some $\Gamma \subseteq \Pi_\beta(C)$. Choose $\hat{p} \leq p$ so that for some $b \in \Gamma$,
[\hat{p}] \leq b. By Lemma 8 of §1, there exists q compatible with \hat{p} with \(|q|(F) < \beta + 1\) and for any r with \(|r|(F) < \beta\), if r and q are compatible then r and \hat{p} are compatible. This q works since if [q] \not\leq b then there exists q_0 \geq q with [q_0] \leq -b. Since -b \in \Sigma_\beta(C) by induction there is q_1 compatible with q_0 with \(|q_1|(F) < \beta\) and [q_1] \leq -b. But then q_1 would be compatible with \hat{p}, contradicting [\hat{p}] \leq b.

Now if X = \omega^\omega, for example, the lemma shows that \mathcal{B} cannot be generated by a set of size less than the continuum in fewer than \alpha steps. For suppose D \subseteq \mathcal{B} has cardinality less than |\omega^\omega|, then there exists F \subseteq \omega^\omega with X - F \neq \emptyset and D \subseteq \Sigma_1([p]: |p|(F) = 0). Let \beta < \alpha, z \in X - F, and \mathcal{E} \subseteq \{\phi\} with |\mathcal{E}| = \beta (where T is the normal \alpha-tree used in the definition of \mathcal{P}_{\alpha}(\phi,X)).

[\{(s,z)\}] is not in \Sigma_\beta(D). Because if it were it would be in \Sigma_\beta(C) and so by the lemma there exists q with \(|q|(F) < \beta\) and [q] \subseteq [\{(s,z)\}]. But since |\mathcal{E}| = \beta and z \not\in F we know (s,z) \not\in q. Thus there are n (and m) such that q \cup \{(s^n,z)\} (q \cup \{(s^n,z,m)\} in case |\mathcal{E}| = 1) is in \mathcal{P}, but this is a contradiction.

Next we show \mathcal{B} is countably generated in \alpha steps. Let \hat{\mathcal{E}} = \{[p]: |p|(\phi) = 0\}.
**Claim.** For all $x \in X$ and $s \in T - \{\phi\}$ if $|s|_T = \beta > 1$ then $\{(s,x)\}$ is in $\Pi_\beta(\mathcal{C})$.

**Proof.**

If $|s|_T = 1$ then $\{(s,x)\} = \Pi\{-\{(s^n,x)\} : n,m \in \omega\}$.

If $|s| > 1$ then $\{(s,x)\} = \Pi\{-\{(s^n,x)\} : n \in \omega\}$. For $A \in B$, $-A = \{p \in P : [p] \wedge A = \emptyset\}$. If $(s,x) \in p$ then $[p] \wedge [\{(s^n,x)\}] = \emptyset$ for all $n$. On the other hand if $[p] \wedge [\{(s^n,x)\}] = \emptyset$ for all $n$ then easily $(s,x) \in p$.

Now for any $p \in P [p] = \Pi\{\{(s,x)\} : (s,x) \in p\}$, so $[p] \in \Sigma_\alpha(\mathcal{C})$. For any $A \in B$ $A = \Sigma\{[p] : p \in A\}$ so $A \in \Sigma_\alpha(\mathcal{C})$. Thus $K(B) \leq \alpha$.

We are now ready to consider the case of $\alpha = \omega_1$.

Let $P = \bigcup_{\alpha < \omega_1} P_\alpha(\emptyset, \omega)$. Now the complete boolean algebra associated with $P$ does take $\omega_1$ steps to close (for suitable generators), however $P$ is not countably generated. So we do as follows: Let $(x_\alpha : \alpha < \omega_1)$ be any set of $\omega_1$ distinct elements of $\omega$. Let $*: \omega < \omega \times \omega < \omega + \omega$ be a 1-1 map. Let $T_\alpha$ be the normal tree of rank $\alpha$ used in the construction of $P_\alpha = P_\alpha(\emptyset, \omega)$. Any $G$ which is $P_\alpha$-generic is determined by $G \wedge \{(s,t) \in P_\alpha : |s|_T = 0$ and $t \in \omega < \omega\}$. That is a map $y$ from $T_\alpha^* = \{s \in T_\alpha : |s|_{T_\alpha} = 0\}$ to $\omega < \omega$. Now imagine $G$ $P$-generic and let $y_\alpha : T_\alpha^* \to \omega < \omega$ be the
maps determined by $G$. Let $Y = \{((s,t))^{x_\alpha}: y_\alpha(s) = t \text{ and } \alpha < \omega_1\}$. Form in the generic extension $P_2(\omega^\omega - Y, \omega^\omega) = Q$ (in both cases we mean $\omega^\omega$ formed in the ground model). The partial order we are interested in is $R = P \times Q$. $P \times Q = \{(p,q): p \in P \text{ and } p \parallel\neg\neg\neg q \in Q\}$. $(\beta, q) \geq (p, q)$ iff $(\beta \geq p \text{ and } q \geq q)$. $p\parallel\neg\neg\neg q \in Q$ just in case whenever $((n), ((s,t))^{x_\alpha})$ is in $q$ then $(s,t) \in p(\alpha)$. Now let $\mathcal{B}$ be the complete boolean algebra associated with $R$. Since $R$ has the countable chain condition so does $\mathcal{B}$.

**Claim:** $\mathcal{B}$ is countably generated

**Proof.**

The idea is that once you know what the real is gotten by $Q$ you know all the reals gotten by $P$--and hence everything. Let $C = \{[\phi,q]: |q|\phi) = 0\}$. Then $C$ is countable and generates $\mathcal{B}$.

For $C \subseteq \omega^\omega$ and $(p,q) \in R$ define $|(p,q)|C) = \max \{|s|_{T_\alpha} : \text{there exists } x \not\in C, (s,x) \in p(\alpha) \text{ and } \alpha < \omega_1\}$

**Lemma 11.** Given $F \subseteq \omega^\omega$ $\forall p \in R \forall \beta \geq 1 \exists \beta, q \in R$ compatible with $p, |\beta|F) < \beta + 1$ and $\forall q|q|(F) < \beta$ (if $\beta, q$ compatible then $p, q$ are compatible).
Proof.
This is proved similarly to Lemma 8. Given
\( p = \langle p_0, p_1 \rangle \) extend each \( p_0(a) \leq p_0^1(a) \) as in Lemma 8,
then take \( \beta = \langle \beta_0, \beta_1 \rangle \), \( \beta_1 = p_1 \), \( \beta_0(a) = \{ \langle s, x \rangle \in p_0^1(a) : \, |s| < \beta + 1 \text{ or } x \in C \} \). Note that \( \beta_0 \models \langle \beta_1 \in Q \rangle \)
because requirements in \( Q \) are decided by rank zero
condition in \( \mathcal{P} \).

From this lemma it is easily shown as before that
\( K(\mathcal{B}) \geq \omega_1 \). Since \( \mathcal{B} \) is countably generated and has the
countable chain condition we have \( K(\mathcal{B}) \leq \omega_1 \), hence
\( K(\mathcal{B}) = \omega_1 \).
This ends the proof of the theorem. \( \square \)

For any \( \sigma \)-complete boolean algebra \( \mathcal{B} \) the Sikorski-Loomis
theorem ([25], p. 93) says that \( \mathcal{B} \) is isomorphic to a
\( \sigma \)-field of subsets of some \( X \) modulo a \( \sigma \)-ideal of
subsets of \( X \).

**Theorem 12.** (Kunen) \( \forall \alpha \leq \omega_1 \exists X, R \) with \( R \subseteq P(X) \) such
that \( K(R) = \alpha \).

**Proof.**
By the Sikorski-Loomis theorem and Theorem 9 we can find
\( \hat{R}, X, \) and \( I \) with \( \hat{R} \subseteq P(X)/I \) where \( I \) is a \( \sigma \)-ideal and
\( \alpha \) is the least ordinal such that \( \hat{R}_\alpha = \hat{R}_{\omega_1} \). Define
\( R \subseteq P(X) \) by \( (A \in R \iff A/I \in \hat{R}) \). It is easily shown
by induction on \( \beta \leq \omega_1 \) that \( (A \in R_\beta \iff A/I \in \hat{R}_\beta) \).
Hence we have \( K(R) = \alpha \). \( \square \)
Let $\mathcal{B}_M$ be the complete boolean algebra $\text{Borel}(\mathcal{P}(\omega))$ modulo the ideal of meager sets.

**Theorem 13.** For any $\alpha, 1 \leq \alpha < \omega_1$, there is a countable $C \subseteq \mathcal{B}_M$ which is closed under finite conjunction and complementation so that $\alpha$ is the least ordinal such that $\Sigma_\alpha(C) = \mathcal{B}_M$.

**Proof.**

Let $x \in \omega^\omega$ be arbitrary and $\mathcal{B}$ be the complete boolean algebra associated with $\mathcal{P}_\alpha(\emptyset,\{x\})$. Note that if $|p|(\emptyset) = 0$ then $-[p] = \Sigma\{q \mid |q|(\emptyset) = 0 \text{ and } q \text{ is incompatible with } p\}$. Let $C$ be the closure of $\{[p] \mid |p|(\emptyset) = 0\} = \hat{C}$ under finite boolean combinations. Note that since $\hat{C}$ is closed under finite intersections and $-[p]$ is in $\Sigma_i(\hat{C})$ for any $p$ in $\hat{C}$, we have that $\Sigma_\beta(C) = \Sigma_\beta(\hat{C})$ for all $\beta \geq 1$. By Lemma 10 $\alpha$ is the least such that $\Sigma_\alpha(\hat{C}) = \mathcal{B}$. Since $\mathcal{P}_\alpha(\emptyset,\{x\})$ is countable and separative, $\mathcal{B}$ is separable and nonatomic and hence isomorphic to $\mathcal{B}_M$. □

**Remark:** The theorem above is false for $\alpha = \omega_1$ since for any countable $C$ which generates $\mathcal{B}_M$, at some countable stage every clopen set is generated and after one more step all of $\mathcal{B}_M$. 
§3 Countably generated Borel hierarchies

A set $X \subseteq 2^\omega$ is called a Luzin set iff $X$ is uncountable and for every meager $M$, $M \cap X$ is countable. The analogous definition with measure zero in place of meager is of a Sierpinski set [30]. For $I$ a $\sigma$-ideal in $\text{Borel}(2^\omega)$ say $X$ is $I$-Luzin iff \[ \forall A \in \text{Borel}(2^\omega) \left( |A \cap X| < 2^0 \iff A \in I \right) \]. The following theorem was first proved by Luzin [12] assuming $I$ is the ideal of meager sets and CH.

Theorem 14.

(MA) If $I$ is an $\omega_1$ saturated $\sigma$-ideal in $\text{Borel}(2^\omega)$ containing singletons then there exists an $I$-Luzin set.

Proof.

Let $\kappa = |2^\omega|$, \( \{A_\alpha : \alpha < \kappa\} = I \), and

\( \{B_\alpha : \alpha < \kappa\} = \text{Borel}(2^\omega) \) - $I$ each set repeated $\kappa$-many times. Choose $x_\alpha$ for $\alpha < \kappa$, so that for every $\alpha$, $x_\alpha$ is in $B_\alpha - (\bigcup \{A_\beta : \beta < \alpha\} \cup \{x_\beta : \beta < \alpha\})$. Clearly if this can be done then $X = \{x_\alpha : \alpha < \kappa\}$ is $I$-Luzin. If $\kappa = \omega_1$ then it is trivial, and if MA then this follows from Lemma 1, p. 158 of Martin-Solovay [14].

The next theorem was proved by Poprougenko [19] and Sierpinski (see [29]).
Theorem 15. If \( X \subseteq 2^\omega \) is a Luzin set then \( \text{ord}(X) = 3 \).

Proof.
Since every Borel set \( B \) has the property of Baire, \( B = G \triangle M \) where \( G \) is open and \( M \) is meager. But \( M \triangle X \triangle F \) is countable hence \( F_0 \), so \( B \triangle X = (G \triangle F) \triangle X \) showing \( \text{ord}(X) \leq 3 \). Now choose \( s \in 2^{<\omega} \) so that \( [s] \triangle X \) is uncountable and dense in \( [s] \). If \( D \subseteq [s] \triangle X \) is countable and dense in \( [s] \) then \( D \not\subseteq G \triangle X \) for all \( G \in G_\delta \), so \( \text{ord}(X) \geq 3 \).

A modern example of a Luzin set arises when one adds an uncountable (in \( M \)) number of product generic Cohen reals \( X \) to \( M \) a countable transitive model of ZFC. \( M[X] \models "X \text{ is a Luzin set}" \). See also Kunen [10] for more on Luzin sets and MA.

In contrast to the boolean algebras Speilrajn [29] showed:

Theorem 16. If \( X \subseteq 2^\omega \) is a Sierpinski set then \( \text{ord}(X) = 2 \).

Proof.
The proof is similar except note that any measurable set is the union of an \( F_\sigma \) set and a set of measure zero.
The following theorem generalizes these classical results using a lemma of Silver (see [14], p. 162) that assuming MA every \( X \subseteq 2^\omega \) with \(|X| < |2^\omega|\) is a Q set, i.e. every subset of \( X \) is an \( F_\sigma \) in \( X \).

Theorem 17. (MA). There are uncountable \( X,Y \subseteq 2^\omega \) such that \( \text{ord}(X) = 3 \) and \( \text{ord}(Y) = 2 \).

Proof.
Let \( X \) be I-Luzin where \( I \) is the ideal of meager Borel sets. For any meager set \( M \) choose \( F \) a meager \( F_\sigma \) with \( M \subseteq F \). By Silver's Lemma there exists \( F_0 \) an \( F_\sigma \) set such that \( F_0 \cap F \cap X = M \cap F \cap X = M \cap X \). Thus every meager set intersected with \( X \) is an \( F_\sigma \) set intersected with \( X \) and this shows as before \( \text{ord}(X) = 3 \). For \( I \) the ideal of measure zero sets analogous arguments work. 

After I had shown that it is consistent with ZFC that \( \forall \alpha \leq \omega_1 \exists X \subseteq \omega^\omega \ \text{ord}(X) = \alpha \), Kunen showed that in fact CH implies \( \forall \alpha < \omega_1 \exists X \subseteq \omega^\omega \ \text{ord}(X) = \alpha \). The following theorem sharpens his result slightly.

Theorem 18. If there exists a Luzin set, then for any \( \alpha \) such that \( 2 < \alpha < \omega_1 \) there is an \( X \subseteq 2^\omega \) such that \( \text{ord}(X) = \alpha \).
Proof.

Let $Y$ be a Luzin set with the property that for every Borel set $A \subseteq 2^{\omega}$ ($A \cap Y$ is countable iff $A$ is meager). Such a set always exists if a Luzin set does. By Theorem 13 there is a $C \subseteq \mathcal{B}_M$ countable such that $C$ generates $\mathcal{B}_M$ in exactly $\alpha$ steps and $C$ is closed under finite Boolean combinations. Let $C = \{[C_n]: n \in \omega\}$ where the $C_n$ are Borel subsets of $2^{\omega}$ and $[C_n]$ is the equivalence class modulo meager of $C_n$. For $x, y \in 2^{\omega}$ define $x \sim y$ iff for all $n < \omega$ ($x \in C_n$ iff $y \in C_n$). We claim that for each $x \in 2^{\omega}$ the $\sim$ equivalence class of $x$ is meager.

Note that any element of the $\sigma$-algebra generated by $\{C_n: n < \omega\}$ is a union of $\sim$ equivalence. If some $\sim$ equivalence class $E$ is not meager, then there are $K_0$ and $K_1$ disjoint nonmeager Borel sets such that $E = K_0 \cup K_1$. Since $\{[C_n]: n < \omega\}$ generates $\mathcal{B}_M$ there are $L_0$ and $L_1$ in the $\sigma$-algebra generated by $\{C_n: n < \omega\}$ such that $[L_0] = [K_0]$ and $[L_1] = [K_1]$. For some $i$, $L_i$ is disjoint from $E$, but then $L_i$ is meager, contradiction. By shrinking $Y$ if necessary we may assume that for all $x, y \in Y$ ($x = y$ iff $x \sim y$). Let $R = \{C_n \cap Y: n < \omega\}$, then $R_1$ contains every countable subset of $Y$. It is easily seen that $K(R) = \alpha$, so by Proposition 2, we are done. ■
Theorem 19. (MA) For any $\alpha < \omega_1$ there is an $X \subseteq \omega^\omega$ such that $\alpha \leq \text{ord}(X) \leq \alpha + 2$.

Proof.

For $\alpha < \omega_1$ let $P_\alpha$ be the partial order $P_\alpha(\emptyset, \omega^\omega)$. Let $T_\alpha$ be the normal tree of rank $\alpha$ used in the definition of $P_\alpha$. $T_\alpha^* = \{ s \in T_\alpha : |s|_{T_\alpha} = 0 \}$. If $G$ is $P_\alpha$-generic, then $G$ is completely determined by the real $y_G : T_\alpha^* \rightarrow \omega^\omega$ defined by $y_G(s) = t$ iff $\{(s,t)\} \in G$. Each condition $p \in P_\alpha$ can be thought of as a statement about $y_G$. Let $C_p = \{ y \in \omega^\omega : y \text{ codes a map } \hat{y} : T_\alpha^* \rightarrow \omega^\omega \text{ and } p(\hat{y}) \text{ is true} \}$. It is easily seen that for any $p \in P_\alpha$ there is $\beta < \alpha$ such that $C_p$ is $\Pi^0_\beta$.

Lemma 20. If $B_\alpha$ is the complete boolean algebra associated with $P_\alpha$ and $X_\alpha$ is $\omega^\omega$ with the topology generated by basic open sets $\{ C_p : p \in P_\alpha \}$, then $B_\alpha$ is isomorphic to the boolean algebra of regular open subsets of $X_\alpha$.

Proof.

Given $A \subseteq X_\alpha$ a regular open set let $D_A = \{ p \in P_\alpha : C_p \subseteq A \}$. The map $A \rightarrow D_A$ is an isomorphism.

Define $I_\alpha$ to the $\sigma$-ideal generated by $\Pi^0_\alpha$ sets of the form $\omega^\omega - U(C_p : p \in D)$ where $D$ is a maximal antichain in $P_\alpha$.

Lemma 21. $\alpha$ is the least ordinal such that for every Borel $A$ there is a $\Sigma^0_\alpha B$ such that $A \Delta B \in I_\alpha$. 
Proof.

Note first that $I_\alpha$ is the ideal of meager subsets of $X_\alpha$. If $D$ is a maximal antichain in $\mathcal{P}_\alpha$, then $\bigcup\{C_p : p \in D\}$ is open dense in $X_\alpha$, so every element of $I_\alpha$ is meager in $X_\alpha$. If $C$ is closed nowhere dense in $X_\alpha$, then let $Q = \{p \in \mathcal{P} : C_p \cap C = \emptyset\}$. Since $Q$ is open dense in $\mathcal{P}_\alpha$, we can pick $D \subseteq Q$ a maximal antichain. Thus $C \subseteq \omega^\omega - \bigcup\{C_p : p \in D\}$ and every meager subset of $X_\alpha$ is in $I_\alpha$.

Since $A$ is Borel in $X_\alpha$ there is a regular open set $B$ in $X_\alpha$ such that $(A \Delta B) \in I_\alpha$. Let $Q = \{p \in \mathcal{P}_\alpha : C_p \subseteq B\}$. Pick $D \subseteq Q$ an antichain which is maximal with respect to being contained in $Q$. Since $B$ is regular open, $B = \bigcup\{C_p : p \in D\}$, so $B$ is $\Sigma^0_\beta$ in $\omega^\omega$. To see that $\alpha$ is minimal note that for $s \in T_\alpha$ with $|s|_{T_\alpha} = \beta$ there is no $B \Sigma^0_\beta$ in $\omega^\omega$ with $(C(s,x) \Delta B) \in I_\alpha$. □

Now let $X \subseteq \omega^\omega$ be $I_\alpha$-Luzin. Then $\text{ord}(X) \geq \alpha$ since for any $A$ and $B$ Borel in $\omega^\omega$ ($(A \Delta B) \in I_\alpha$ iff $|(A \Delta B) \cap X| < |X|$). But $\text{ord}(X) \leq \alpha + 2$ follows from the fact that for all $B$ in $I_\alpha$ there exists $C$ in $I_\alpha \cap \Sigma^0_{\alpha+1}$ with $B \subseteq C$, just as in the proof of Theorem 17. This concludes the proof of Theorem 19. □
Remarks:
(1) If $V = L$, then using the $\Delta^1_2$ well ordering of $L \cap 2^\omega$ we can get $X \subseteq 2^\omega$ a $\Delta^1_2$ set with $\text{ord}(X) = \alpha$ for any $\alpha \leq \omega_1$. If $X$ is $\Pi^1_1$ (or $\Sigma^1_1$) then $X = A \Delta M$ where $A$ is $\Pi^0_\alpha$ and $M \in I_\alpha$, so $X$ cannot be $I_\alpha$-Luzin.

(2) A finer index can be given to a set $X \subseteq \omega^\omega$ by considering the classical Hausdorff difference hierarchies. A set $C \subseteq \omega^\omega$ is a $\beta - \Pi^0_\alpha$ set iff there exists $D_\gamma \in \Pi^0_\alpha$ for $\gamma < \beta$ such that the $D_\gamma$'s are decreasing and $D_\lambda = \gamma < \lambda D_\gamma$ for $\lambda$ limit and $C = \bigcup (D_\gamma - D_{\gamma+1} : \gamma < \beta$ and $\gamma$ even). It is a theorem of Hausdorff that $\Delta^0_{\alpha+1} = \bigcup (\beta - \Pi^0_\alpha : \beta < \omega_1)$ (see p. 417, 448 [11]).

It is also not hard to show, using a universal set argument, that there exists a properly $\beta - \Pi^0_\alpha$ set for all $\alpha, \beta < \omega_1$. Accordingly define $H(X)$ to be the lexicographical least pair $(\alpha, \beta) \in \omega^2_1$ such that for any Borel set $A$ there exists $B$ a $\beta - \Pi^0_\alpha$ set such that $A \cap X = B \cap X$. If $X$ is a Luzin set (Sierpinski set) then $H(X) = (2, 2)$ ($H(X) = (2, 1)$). It is easily shown that in Theorem 22 $N \models "H(X_{\alpha+1}) = (\alpha + 1, 1)"$. It is not hard to see that for $C$ a countable closed set $H(C) = (1, \alpha)$ where $\alpha$ is the Cantor-Bendixson rank of $C$. 
Theorem 22. It is relatively consistent with ZFC that for any uncountable $X \subseteq 2^\omega$ $\text{ord}(X) = \omega_1$. This can be generalized to show that for any successor ordinal $\beta_0$ such that $2 \leq \beta_0 < \omega_1$, it is consistent that 

$$\{ \beta : \exists X \subseteq 2^\omega \text{ uncountable } \text{ord}(X) = \beta \} = \{ \beta : \beta_0 \leq \beta \leq \omega_1 \}.$$ 

Remark: It is true in the model obtained that for any uncountable separable metric space $X$ the Borel hierarchy on $X$ has length $\omega_1$. This is true, since if $|X| = \omega_1$, then since $|2^\omega| > \omega_2$ and $X$ can be embedded into $\mathbb{R}^\omega$, $X$ must be zero dimensional. But any zero dimensional space can be embedded into $2^\omega$.

To prove Theorem 22 let $M$ be a countable transitive model of ZFC + GCH. Choose $(\alpha_\lambda : \lambda < \omega_2)$ in $M$ so that for all $\beta < \omega_1$ $\{ \lambda : \alpha_\lambda = \beta \}$ is unbounded in $\omega_2$.

Define $P^\gamma$ for $\gamma \leq \omega_2$ by induction $P^0 = P_{\alpha_0}(\phi, 2^\omega \cap M)$,

$$P^{\gamma+1} = P^{\gamma} \ast Q^\gamma$$

where $Q^\gamma$ is a term in the forcing language of $P^\gamma$ denoting $P_{\alpha_\gamma}(\phi, M[G_\gamma] \cap 2^\omega)$ for any $G_\gamma$ $P^\gamma$-generic over $M$, and at limits take the direct limit.

The elements of $Q^\gamma$ can be thought of as terms denoting elements of $2^\omega \cap M[G_\gamma]$ via a natural coding.

Choose such a coding which has the property that for any $p,q \in Q^\gamma$ ($p$ and $q$ are compatible iff there is $n < \omega$ such that $p \upharpoonright n$ and $q \upharpoonright n$ are seen to be compatible).

For $Q \subseteq P$ and $\theta$ a sentence we say that $Q$ decides $\theta$
iff \( \{ p \in \mathcal{P} : \text{there is a } q \in Q \text{ such that } p \geq q \text{ and } (q \vdash \theta \text{ or } q \vdash \neg \theta) \} \) is dense in \( \mathcal{P} \). For any \( H \subseteq 2^\omega \) define \( |p|(H) \) and \( |\tau|(H,p) \) for \( p \in \mathcal{P}^\gamma \) and \( \tau \) a \( \mathcal{P}^\gamma \) term for an element of \( 2^\omega \) by induction on \( \gamma \).

For \( p \in \mathcal{P}^0 = \mathcal{P}_{\alpha_0}(\emptyset, 2^\omega \cap M), \ |p|(H) = \max\{|s|_{\tau_{\alpha_0}} : \exists x \notin H (s,x) \in p\}. \) For \( p \in \mathcal{P}^{\gamma+1}, \ |p|(H) = \max\{|p \upharpoonright \gamma|(H), |p(\gamma)|(H, p \upharpoonright \gamma)\}. \) For \( p \in \mathcal{P}^\lambda \) where \( \lambda \) is a limit, \( |p|(H) = \max\{|p \upharpoonright \gamma| : \gamma < \lambda\}. \)

For any \( \tau, |\tau|(H,p) \) is the least \( \beta \) such that for any \( n \in \omega \) \( \{ q \in \mathcal{P}^\gamma : q \text{ incompatible with } p \text{ or } |q|(H) \leq \beta \} \) decides "\( n \in \tau \)."

\( \mathcal{P}^\omega = \mathcal{P} \) is not a lattice however it does have one similar property:

\textbf{Lemma 23.} Suppose \( G \) is \( \mathcal{P}^\alpha \)-generic over \( M \) and for \( i < n < \omega \) \( q_i \in G \) and \( |q_i|(H) < \beta \), then there is a \( q \in G \) with \( |q|(H) < \beta \) and \( q \leq q_i \) for all \( i < n \).

\textbf{Proof.}

The proof is by induction on \( \alpha \). For \( \alpha = 0 \) or a \( \alpha \) a limit it is easy. So suppose \( \alpha = \beta + 1 \) and \( G = G_\beta \times G_\beta \) where \( G_\beta \) is \( \mathcal{P}^\beta \)-generic over \( M \). Find \( \Gamma \subseteq G_\beta \) finite so that for any \( q \in \Gamma \) with \( |q|(H) < \beta \) and for any \( i \) and \( j \) less than \( n \) if \( (s, \tau) \in q_i(\beta) \) and \( (s^\pi k, \tilde{\tau}) \in q_j(\beta) \)
(or \((s^k, t) \in q_j(\beta)\) where \(t \in 2^{<\omega}\)), then there is \(r \in \Gamma\) such that \(r \models \neg \tau(t \in \tau')\). By induction there is \(q\) in \(G_\beta\) such that \(|q(H)| < \beta\), for all \(q \in \Gamma\) \(q \leq q\), and for all \(i < n\) \(q \geq q_i \triangleright \beta\). Define \(q(\beta)\) to be equal to \(\bigcup\{q_i(\beta) : i < n\}\).

**Lemma 24.** Given \(P_0\), a countable subset of \(P^\alpha\) and \(Q_0\), a countable set of \(P^\alpha\) terms for elements of \(2^\omega\), there exists \(H\) countable such that for every \(p \in P_0\) and \(\tau \in Q_0\) \(|p|_H = |\tau|_H = 0\).

**Proof.**
This is easy using c.c.c. of \(P^\alpha\). \(\square\)

Let \(|p| = p(H)\) and \(|\tau|_H(p) = |\tau|_H(p, \phi)\), for some fixed \(H\). 

**Lemma 25.** For each \(p \in P^\alpha\) and \(\beta\) there exists \(\hat{p} \in P^\alpha\) compatible with \(p\), \(|\hat{p}| < \beta + 1\), and for every \(q \in P^\alpha\) with \(|q| < \beta\), if \(\hat{p}\) and \(q\) are compatible, then \(p\) and \(q\) are compatible.

**Proof.**
The proof is by induction on \(\alpha\). For \(\alpha = 0\) this is just Lemma 8 of §1. For \(\alpha\) limit it is easy. From now on assume the Lemma is true for \(\alpha\).

Define for \(x, y \in 2^\omega\), \(x <_\lambda y\) iff \(\exists n \forall m < n(x(m) = y(m)\) and \(x(n) < y(n)\)). This is the
lexicographical order. For $C \subseteq 2^\omega$ a nonempty closed set let $x_C$ be the lexicographically least element of $C$.

Claim 1. Let $\mathcal{C}$ be a term in $\mathcal{P}^\alpha$ and $p_0 \in \mathcal{P}^\alpha$ with $|p_0| < \beta + 1$ such that $p_0 \models "\mathcal{C}"$ is a nonempty closed subset of $2^\omega$. Suppose for every $G \mathcal{P}^\alpha$-generic with $p_0 \in G$, and $s \in 2^{<\omega}(M[G]) \models "[s] \cap \mathcal{C} \neq \emptyset"$ iff

$\exists q \in G, |q| < \beta$, and $q \models "[s] \cap \mathcal{C} = \emptyset"$). Then $|x_C|(p_0) < \beta + 1$.

Proof.

First we show that given any $p \in \mathcal{P}^\alpha$ with $p \geq p_0$, if $s \in 2^{<\omega}$, $p \models "[s] \cap \mathcal{C} \neq \emptyset"$ then there exist $\hat{p} \in \mathcal{P}^\alpha$ compatible with $p$, $|\hat{p}| < \beta + 1$, and $\hat{p} \models "[s] \cap \mathcal{C} \neq \emptyset"$.

Let $p'$ be as from Lemma 25 for $p$. By using Lemma 23 obtain $\hat{p}$ compatible with $p, \hat{p} \geq p'$, $\hat{p} \geq p_0$, and $|\hat{p}| < \beta + 1$. I claim $\hat{p} \models "[s] \cap \mathcal{C} \neq \emptyset"$. Suppose not then there exists $G \mathcal{P}^\alpha$-generic, $\hat{p} \in G$, and $M[G] \models "[s] \cap \mathcal{C} = \emptyset"$. So there exists $q \in G, |q| < \beta$, and $q \models "[s] \cap \mathcal{C} = \emptyset"$. But then since $q$ is compatible with $\hat{p}$ it is compatible with $p'$ and hence with $p$, contradiction. In order to show $|x_C|(p_0) < \beta + 1$ it suffices to show for every $p \geq p_0$ and $n \in \omega$ there exist $\hat{p} \in \mathcal{P}^\alpha$ compatible with $p$, $|\hat{p}| < \beta + 1$, and there exists $s \in 2^n$ such that $\hat{p} \models "x_C \upharpoonright n = s"$. So given $p$ and $n$ find
r ≥ p and s ∈ 2^n such that r |- "x_C ∨ n = s". We have just shown there exists f compatible with r with |f| < β + 1 and f |- "[s] ∩ C ≠ ∅". Let G be Pα-generic containing r and f. For each t ∈ 2^{m+1} with m + 1 ≤ n and for all k < m (t(k) = s(k) and t(m) < s(m)) choose q_t ∈ G with |q_t| < β and q_t |- "[t] ∩ C = ∅". (There are only finitely many such t). Choose q ∈ G with |q| < β + 1, q ≥ f, and q ≥ q_t for each such t. (q exists by Lemma 23). Then q |- "x_C ∨ n = s". 

For p and q compatible define p ∪ q |- "θ" to mean that for every r, if r ≥ p and r ≥ q then r |- "θ". For τ a Pα term for an element of 2^ω and p ∈ Pα, define C(τ, p) a Pα term so that for any G which is Pα-generic (it need not contain p) C^G(τ, p) = ∩{D_τ: there exist q ∈ G, |q| < β, |τ|(q) < β, q |- "τ ∈ D_τ"}, p and q are compatible, and p ∪ q |- "τ ∈ D_τ"}. D is a universal Π₀¹ subset of 2^ω × 2^ω (∀K ∈ Π₀¹ ∃x ∈ 2^ω K = D_x = {y: (x,y) ∈ D}).

Claim 2. Let p be given by Lemma 25 for p ∈ Pα (i.e. for all q ∈ Pα if |q| < β, then if q and p are compatible then q and p are compatible). Then p and
C(τ,p) satisfy the hypothesis of Claim 1 for p₀ and C.

Proof.
Suppose M[G] |= "[s] ∩ C(τ,p) = ∅". By compactness there exists n < ω, qᵢ ∈ G, τᵢ for i < n with |qᵢ| < β and |τᵢ|(qᵢ) < β so that p ⋃ qᵢ |− "τ ∈ Dᵢ" and M[G] |= "∩{Dᵢ : i < n} ∩ [s] = ∅". Let τ̂ be a term for an element of 2^ω so that D̂ = ∩{Dᵢ : i < n} and q ∈ G with q ≥ qᵢ for i < n and |q| < β. (τ̂ can be chosen so that |τ̂|(q) < β assuming some nice properties of D).

Since q and p are compatible, q and p are compatible and q ⋃ p |− "τ ∈ D̂". Since M[G] |= "D̂ ∩ [s] = ∅" by compactness there exists m ∈ ω so that if t = τ̂G then for every x ≥ t, x ∈ 2^ω Dₓ ∩ [s] = ∅. Since |

The fact that p |− "C(τ,p) \neq ∅" follows from this since if not there exists q compatible with p, |q| < β, and q|− "[∅] ∩ C(τ,p) = ∅". But then q is compatible with p contradiction.

We now return to the proof of the α + 1 step of Lemma 25.

Assume p ∈ P^α⁺ is given with the following property:
(*) there exists $s_\tau \in 2^{<\omega}$ for each $\tau$ such that there exist $s$ with $(s,\tau) \in P(\alpha)$ and $|s| \geq 1$. And these $s_\tau$ have the property that $\emptyset \models \langle s_\tau \subseteq \tau \rangle$ and whenever $(s,\tau), (s',i,\hat{\tau}) \in p(\alpha)$ (or $(s',i,\tau) \in p(\alpha)$ where $\tau \in 2^{<\omega}$) $s_\tau$ and $s'_\tau$ are incompatible (or $s_\tau$ and $\tau$ are incompatible).

The set of $p$'s with this property is dense in $P^{\alpha+1}$ so it is enough to prove the Lemma 25 for them. Let $(s_i,\tau_i)$ for $i < n$ be all $(s,\tau) \in p(\alpha)$ with $|s| \geq 1$ and let $\bar{\tau} = (\tau_0, \tau_1, \ldots, \tau_{n-1})$ (where $(\tau_0, \ldots, \tau_{n-1}) : (2^\omega)^n \to 2^\omega$ is some recursive coding). Let $\hat{p}_{\alpha}$ be as given from Lemma 25 for $p_{\alpha}$. Let $\bar{\tau}$ be the lexicographical least element of $C(\bar{\tau}, p_{\alpha})$. By Claim 1 and 2 $|\bar{\tau}| (\hat{p}_{\alpha} \cup q_{\alpha}) < \beta + 1$. Now let $\hat{p}(\alpha) = \{(s,t) \in p(\alpha) : |s| = 0\} \cup \{(s_i,\tau_i) : i < n\} (\bar{\tau}^\lambda = (\tau_0^\lambda, \ldots, \tau_{n-1}^\lambda)).$ Since $\emptyset \models \langle C(\bar{\tau}, p_{\alpha}) \rangle$ is included in $\bigcap_{i \in \omega} [s_i,\tau_i]$, $\hat{p}$ is a condition, $\hat{p}$ and $p$ are compatible, also $|\hat{p}| < \beta + 1$. Now suppose $q \in P^{\alpha+1}$ compatible with $\hat{p}, |q| < \beta$, and $q$ and $p$ are not compatible. Let $G$ be $P_{\alpha}$-generic with $\hat{p}_{\alpha}$ and $q_{\alpha}$ elements of $G$ and $M[G] \models \langle \hat{p}(\alpha)$ and $q(\alpha)$ are compatible$\rangle$. If we think of $p(\alpha)$ as a statement about $\bar{\tau}$ i.e. $p(\alpha)(\bar{\tau})$ then $\hat{p}(\alpha) = p(\alpha)(\bar{\tau})$. Since $p$ and $q$ are incompatible but $p_{\alpha}$ and $q_{\alpha}$ are compatible $(p_{\alpha} \cup q_{\alpha}) \models \langle \hat{p}(\alpha)$ and
q(α) are incompatible". D(τ) \equiv "p(α)(\overline{τ}) and q(α) are incompatible" is a \( \Pi_1^0 \) statement with parameters from q(α) about \( \overline{τ} \). Thus we conclude that M[G] \models "p(α)(\overline{τ}) and q(α) are incompatible", contradiction. This concludes the proof of Lemma 25.

From now on let \( \mathcal{P} = \mathcal{P}^{\omega_2} \).

Lemma 26. Suppose \(|τ| = 0\), B(ν) is a \( \Sigma_β^0 \) predicate, \( β \geq 1 \), with parameters from M, and \( p \in \mathcal{P} \) is such that \( p \vdash "B(τ)" \); then there exists \( q \in \mathcal{P} \) compatible with \( p \), \(|q| < β \) and \( q \vdash "B(τ)" \).

Proof.
The proof is by induction on \( β \).

Case 1. \( β = 1 \).
Suppose \( p \vdash "\exists n R(x \upharpoonright n, τ \upharpoonright n)" \) for \( R \) recursive and \( x \in M \). Let \( G \) be \( \mathcal{P} \)-generic with \( p \in G \). Choose \( n ∈ ω \) and \( s ∈ 2^n \) so that \( M[G] \models "R(x \upharpoonright n, τ \upharpoonright n) and τ \upharpoonright n = s" \).
Choose \( q ∈ G \) with \(|q| = 0 \) and \( q \vdash "τ \upharpoonright n = s" \).

Case 2. \( β \) is a limit ordinal.
If \( p \vdash "\exists n B(n, τ)" \) then \( \exists \hat{p} \geq p \ \hat{p} \vdash "B(n_0, τ)" \) and \( B(n_0, ν) \ \Sigma_γ^0 \) for \( γ < β \), so apply induction hypothesis to \( \hat{p} \).

Case 3. \( β + 1 \).
Suppose \( p \vdash "\exists n B(n, τ)" \) where \( B(n, ν) \) is
with parameters from $M$. Choose $r \geq p$ and $n_0 \in \omega$ so that $r \models "B(n_0, \tau)"$. By Lemma 25 there is $q$ compatible with $r$, $|q| < \beta + 1$, and for every $s$, $|s| < \beta$, if $q$ and $s$ are compatible, then $r$ and $s$ are compatible. $q \models "B(n_0, \tau)"$ because if not then there is $q' \geq q$ such that $q' \models "B(n_0, \tau)"$, and so by induction there is $s$ with $|s| < \beta$ compatible with $q'$ and $s \models "B(n_0, \tau)"$; but then $s$ is compatible with $r$, contradiction.

Now let us prove the first part of Theorem 22. Let $G$ be $\mathbb{P}$-generic over $M$. We claim $M[G] \models "for every $X \subseteq 2^\omega$ and $\alpha < \omega_1$ if $|X| = \omega_1$ then $\text{ord}(X) \geq \alpha + 1.""$. But since any such $X$ is in some $M[\mathbb{P}_\beta]$ for $\beta < \omega_2$, we may as well $X \in M$, $\alpha_0 = \alpha + 1$, and we must show $M[G] \models "\text{ord}(X) \geq \alpha + 1."$. Let $G_{\alpha_0}$ be the $\Pi^0_{\alpha_0}$ set created by $G \cap \mathbb{P}_{\alpha_0}(\phi, 2^\omega \cap M)$. Suppose that $M[G] \models "\text{there is } K \text{ a } \Pi^0_{\beta_0} \text{ set such that } K \cap X = G_{\alpha_0} \cap X."$. Let $\tau$ be a term for the parameter of $K$. Choose $p \in G$ such that $p \models "\forall z \in X (x \in K \iff z \in G_{\alpha_0})."$. By Lemma 24 there exists $H$ in $M$ countable so that $|\tau|(H, \phi) = |p|(H) = 0$. Let $z \in X - H$. Define $\hat{p} \in \mathbb{P}$ by $\hat{p}(0) = p(0) \cup \{(0), z\}$ and $\hat{p}(\alpha) = p(\alpha)$ for $\alpha > 0$. Since $\hat{p}$ says $z \in G_{\alpha_0}$, $p \models "z \in K."$. By Lemma 26 there exists $q$ compatible with $\hat{p}$, $|q|(H) < \beta$, and $q \models "z \in K."$. By Lemma
there exists \( \hat{q} \) with \( |\hat{q}|(H) < \beta, \hat{q} \geq q, \) and \( \hat{q} \geq p. \)

Since \( |(0)|_T^{\alpha_0} = \alpha, ((0),z) \notin \hat{q}(0) \), there exists \( m \in \omega \) such that \( r \) defined by \( r(0) = q(0) \cup \{(0,m),z\} \) and \( r(\alpha) = \hat{q}(\alpha) \) for \( \alpha > 0 \) is a condition. But this is a contradiction since \( r \models ' (z \in G(\nu) \iff z \in K) \) and \( z \in K \) and \( z \notin G(\nu) \).

Now we prove the second sentence of Theorem 22.

Let \( X = \bigcup \{ X_\alpha : \beta_0 \leq \alpha < \omega_1 \) and \( \alpha \) a successor\} where each \( X_\alpha \) is a set of \( \omega_1 \) product generic Cohen reals.

Let \( M_0 = M[X] \). Define in \( M_0 \) the partial order \( P^Y \) for \( Y \leq \omega_2 \) so that \( P^{Y+1} = \bigvee P^{Y} \) where \( Q_Y \) is a term denoting:

**Case 1.** \( P_{\beta_0}(\phi,M_0[G_Y] \wedge 2^\omega) \) or

**Case 2.** \( P_\beta(Y_\gamma,X_\beta \cup F) \) where \( Y_\gamma \) is a Borel subset of \( X_\beta \) in \( M_0[G_Y] \) and \( F = \{ x \in 2^\omega : x \) eventually zero\}.

Case 1 is done cofinally in \( \omega_2 \) and Case 2 is done in such a way as to insure:

\( M_0[G_{\omega_2}] \models ' \)For every successor ordinal \( \beta \) with \( \beta_0 \leq \beta < \omega_1 \) and \( Y \) Borel in \( X_\beta \) there is a \( \gamma \) such that \( Y = Y_\gamma \)'. First we show that essentially the same arguments as before show that \( M_0[G_{\omega_2}] \models ' \)For every \( X \subseteq 2^\omega \) uncountable \( \text{ord}(X) > \beta_0 \)'. This will not use that the \( X_\alpha \) are made up of Cohen reals, hence any of the intermediate models would serve as the ground model. So
suppose Case 1 occurs on the first step, $Y \in M_0$ is uncountable, $\beta_0 = \gamma + 1$, and $M_0[G_{\omega_2}] \models "Y \cap G(\emptyset) = Y \cap J$ for some $J \in {}^\omega_2 \gamma"$. Given $L \subseteq \omega_2$ define $P_\alpha^\omega$ as follows.

For $\alpha \in L$:

Case 1. $P_\alpha^{\alpha+1} = P_\alpha^\alpha \ast P_{\beta_0}(\phi, M[G_\alpha] \cap 2^\omega)$ where $G_\alpha$ is $P_\alpha^\alpha$-generic over $M_0$.

Case 2. $P_\alpha^{\alpha+1} = P_\alpha^\alpha \ast P_\beta(Y_\alpha - F, X_\beta \cup F)$ (where we assume $L$ has the property that when Case 2 happens for $\alpha \in L$ then $Y_\alpha$ is a Borel subset of $X_\beta$ coded by some term $\tau_\alpha$ in $P_\alpha^\beta$).

For $\alpha \notin L$:

$P_\alpha^{\alpha+1} = P_\alpha^\alpha \ast 1$ (singleton partial order).

Note that by using c.c.c. of $P_\omega^{\omega_2}$ we can find $L \subseteq \omega_2$ countable, so that the Borel code for the above $J$ is a $P_\alpha^{\omega_2}$ term and $L$ has the property mentioned under Case 2. For $\alpha$ a limit $P_\alpha^\omega$ is the direct limit of $(P_\beta^\beta : \beta < \alpha)$.

Lemma 27. If $N \supseteq M$ is a forcing extension and $G$ is $P_\beta(\phi, N \cap 2^\omega)$ generic over $N$ then $G \cap P_\beta(\phi, M \cap 2^\omega)$ is $P_\beta(\phi, M \cap 2^\omega)$ generic over $M$.

Proof.
It is enough to show that for any $\Delta \in M$ dense in
Let $N$ be an extension of $M$ via a partial order $Q$. Given $p \in P_\beta(\phi, 2^\omega \land N)$ (a term in the forcing language of $Q$) suppose $\exists q \in Q \ q \models \forall r \in \Delta \ r$ and $p$ are incompatible. View $p$ as being coded up in some natural way by a single real in $2^\omega \land N$. Then we can find $\hat{p} \in P_\beta(\phi, 2^\omega \land M)$ so that $\forall n < \omega \ \exists \hat{q} > q \ \hat{q} \models \forall r \in \Delta \ r$ and $\hat{p}$ are compatible. But compatibility is witnessed by $\hat{p}n$ for some $n < \omega$. Let $\hat{q} > q$ and $\hat{q} \models \forall r \in \Delta \ r$ and $\hat{p}$ are compatible, contradiction.

**Lemma 28.** Suppose $P_0, P_1 \in M$ are partial orders and $\exists \tau$ a term in language of $P_i$ such that $\forall G \ P_i$-generic over $M$, $\tau^G$ is $P_0$-generic over $M$. Then $\forall G \ P_i$-generic over $M$, $M[G]$ is a forcing extension of $M[\tau^G]$.

**Proof.**

This is easier to prove using the cBa approach to forcing. Let $B_i$ for $i = 0, 1$ be the associated cBa to $P_i$ for $i = 0, 1$ and $\hat{\tau}$ a $B_i$ term so that $\forall G \ B_i$-generic $\hat{\tau}^G$ is $B_0$-generic. Define a map $j : B_0 \rightarrow B_1$ by $j(p) = \{ p \in \hat{\tau} \}_{B_1}$. Then $j$ is an isomorphism of $B_0$ onto an $M$-complete subalgebra of $B_1$. Otherwise suppose $\Gamma \subseteq B_0 \setminus \{ \phi \}, \Gamma \in M$ and
Choose \( G \) \( B \)-generic with \( e \cdot f \in G \). Then \( \forall p \in \hat{T} \) \( p \in \hat{T}^G \) and \( \forall p \in \hat{T} \) \( p \notin \hat{T}^G \). But this means \( \hat{T}^G \) is not \( B \)-generic over \( M \) (see Lemma p. 35 Solovay [27]). But now by Lemma 5.2.4 of Solovay-Tennenbaum [26] we are done.

Given any \( G \) \( P^{\omega_2} \)-generic let \( G_L \) be the subset of \( P_L \) generated by the rank zero conditions in \( G \). The two preceding lemmas enable us to prove:

**Lemma 29.** For any \( \alpha \)

if \( G_\alpha \) is \( P^\alpha \)-generic over \( M_0 \) then \( G_L^\alpha \) is \( P^\alpha \)-generic over \( M_0 \).

**Proof.**

This is proved by induction on \( \alpha \). For \( \alpha + 1 \neq L \) it is immediate. For \( \alpha + 1 \in L \) Case 1 is handled by Lemma 27, Lemma 28, and the product lemma. Case 2 is easy as \( P_B(Y_\alpha - F, X_\beta \cup F) \) is the same partial order in either case.

For \( \alpha \) limit ordinal let \( \Delta \in P^\alpha_L \) be dense, we show \( \{ q \in P^\alpha \mid \exists p \in \Delta \ p \leq q \} \) is dense in \( P^\alpha \). If \( q \in P^\alpha \) then \( q \in P^\beta_\beta \) for some \( \beta < \alpha \). Let \( \Delta_\beta = \{ p \beta : p \in \Delta \} \), then \( \Delta_\beta \) is dense in \( P^\beta_\beta \). Hence if \( G_\alpha \) is \( P^\alpha \)-generic with \( q \in G_\alpha \) then since \( G_\beta^L \) is \( P^\beta \)-generic it meets \( \Delta_\beta \)--say at \( p \beta \). But then \( q \) and \( p \) are compatible.
Define for \( H \subseteq 2^\omega \mid \rho \mid(H), \mid \tau \mid(H,p) \) for \( p \in P^\alpha_L \) and \( \tau \) a \( P^\alpha_L \)-term for a subset of \( \omega \) by induction on \( \alpha \).

**Case 1.** \( P^{\alpha+1}_L = P^\alpha \times P^\beta_0 (\phi, M[G^\alpha_L] \triangleleft 2^\omega) \)

\[ |\rho|(H) = \max\{|\rho \gamma| (H)\}, |\rho(\gamma)| (H, p \gamma) \} \] (same as before).

**Case 2.** \( P^{\alpha+1}_L = P^\alpha \times P^\beta (Y_\alpha - F, X_\alpha \cup F) \)

\[ |\rho|(H) = \max\{|\rho \alpha| (H), |\rho \alpha| (H, p \alpha) \} \]

\[ |\tau|(H,p) \] is defined as it was just before Lemma 23.

Lemma 23 is easily proven since in Case 2 we have a lattice.

Lemma 24 is also easily proven if in addition \( H \) is taken with the property that \( \forall x \in H \ \forall \alpha \in L \)

\( \{p: |p|(H) = 0\} \) decides "\( x \in Y_\alpha \)" whenever Case 2 happens at stage \( \alpha \). Lemma 25 can be proven for \( \beta < \beta_0 \) by the same argument in Case 1 and by the argument of Theorem 34 in Case 2. Lemma 26 follows and so does the claim that 

\[ M_0[G_{\omega_2}] = "K \subseteq \{\beta: \beta_0 < \beta < \omega_1\}". \]

Next we show \( M_0[G_{\omega_2}] = "\text{ord}(X_\beta) = \beta \) for each \( \beta \) successor \( \beta_0 < \beta < \omega_1\). \) If not then again we can reduce to some \( L \triangleleft \aleph_2 \) countable; and since each \( X_\alpha \) is present in \( M_0 \), we can relabel \( L \) so that for some \( \beta < \omega_1 \) and \( \beta_1 \) with \( \beta_0 < \beta_1 < \omega_1 \), \( M_0[G_{\beta}] = "\text{ord}(X_{\beta_1}) < \beta_1" \) for \( G_{\beta} \) \( P^\beta \)-generic over \( M_0 \), and on some step before \( \beta \) we force with 

\[ P^\beta_{\beta_1} (\phi, X_{\beta_1} \cup F) \]. Suppose \( X = \{x_\alpha: \alpha < \omega_1\} \) and 

\[ M_0 = M[\{\langle \alpha, x_\alpha \rangle: \alpha < \omega_1\}] \]. Given \( H \subseteq \omega_1 \), \( H \in M \) let
\[ \hat{A} = \{ \langle \alpha, x_\alpha \rangle : \alpha \in H \}. \] Define \( \mathbb{P}_H^\alpha \in M[\hat{A}] \) for each \( \alpha < \hat{\beta} \).

**Case 1** \( \mathbb{P}_H^{\alpha+1} = \mathbb{P}_H^\alpha \times \mathbb{P}_0 (\phi, M[G_H^\alpha] \cup 2^\omega) \).

**Case 2** \( \mathbb{P}_H^{\alpha+1} = \mathbb{P}_H^\alpha \times \mathbb{P}_\beta ((Y_\beta - F) \cup \hat{H}, (X_\beta \cup \hat{H}) \cup F) \)

(assuming \( Y_\alpha \) is a Borel subset of \( X_\beta \) given by the term \( \tau_\alpha \) in forcing language of \( \mathbb{P}_H^\alpha \)).

**Lemma 30.** For any \( \alpha < \beta \) if \( G^\alpha \) is \( \mathbb{P}_H^\alpha \)-generic over \( M_0 \), then \( G_H^\alpha \) is \( \mathbb{P}_H^\alpha \)-generic over \( M[\hat{H}] \).

**Proof.**

The proof is like Lemma 29 except on \( \alpha + 1 \) under Case 2.

\[ \mathbb{P}_1 = \mathbb{P}_\beta (Y_\alpha - F, X_\beta \cup F) \text{ in } M[X][G^\alpha] = M_1 \]

\[ \mathbb{P}_2 = \mathbb{P}_\beta ((Y_\alpha - F) \cup \hat{H}, (X_\beta \cup \hat{H}) \cup F) \text{ in } M[\hat{H}][G_H^\alpha] = M_2. \]

Again suppose \( \Delta \in M_2 \) is dense in \( \mathbb{P}_2 \), we show

\[ \{ p \in \mathbb{P}_1 : \exists q \in \Delta \ q \leq p \} \text{ is dense in } \mathbb{P}_1. \] Given \( p \in \mathbb{P}_1 \), let \( p = r \cup \{ s_n, x_n : n < N \} \) where \( x_n \in X_\alpha - \hat{H}, N < \omega \), and \( r \in \mathbb{P}_2 \). Let \( Q_N \) be the partial order for adding \( N \) Cohen reals.

By the product lemma \( \{ x_n : n < N \} \) is \( Q_N \)-generic over \( M_2 \), and also \( p \in M_2[\{ x_n : n < N \}] \). Hence if \( \forall q \in \Delta \ p \) and \( q \) are incompatible in

\[ \mathbb{P}_3 = \mathbb{P}_\beta ((Y_\alpha - F) \cap (H \cup \{ x_n : n < N \}, (X_\beta \cap H \cup \{ x_n : n < N \}) \cup F) \]

then \( \exists \bar{p} \in Q_N \ | - \) "\( \forall q \in \Delta \ p \) and \( q \) are incompatible in \( \mathbb{P}_3 \)". Choose \( y_n \in F \) for \( n < N \) so that

\[ p_0 = r \cup \{ s_n, y_n : n < N \} \in \mathbb{P}_2 \]

and

\[ \forall m < \omega \exists \bar{p}' \geq \bar{p} \forall n < N : \bar{p}' | - "y_{n m} = x_{n m} \". \] Since

\[ \exists q \in \Delta \ p_0 \] and \( q \) are compatible, then as before \( p \) and
q can be forced compatible by an extension of \( \hat{p} \). So \( p \) and \( q \) are compatible in \( P_3 \) and hence in \( P_1 \).

Lemma 31. Given \( \tau \) a term in forcing language of \( P^\beta \) if \( p \in P^\beta \), \( p \models_{P^\beta} "B(\tau)" \) where \( B(\check{\nu}) \) is a \( \Sigma_1 \) predicate with parameters in \( M[H] \) then \( \exists q \in P^\beta_H \) compatible with \( p \) such that \( q \models_{P^\beta} "B(\tau)" \).

Proof.
Let \( G \) be \( P^\beta \)-generic over \( M_0 \) with \( p \in G \). Then by Lemma 9 \( G_H \) is \( P^\beta_H \)-generic over \( M[H] \). Since \( \Sigma_1 \) sentences are absolute and \( M_0[G] \models "B(\tau)" \) we have \( M[H][G_H] \models "B(\tau)" \). So \( \exists q \in G_H \) \( q \models_{P^\beta_H} "B(\tau)" \). But for any \( G \) \( P^\beta \)-generic containing \( q \), \( M[H][G_H] \models "B(\tau)" \) whence by absoluteness \( M_0[G] \models "B(\tau)" \). We conclude \( q \models_{P^\beta} "B(\tau)" \).

Lemma 32. Given \( H = X - \{z\} \) where \( z \in X_{\alpha+1} \), \( \gamma \leq \hat{\beta} \), \( 1 \leq \beta < \alpha \), \( p \in P^\gamma \) then \( \exists \hat{p} \in P^\gamma_H \) \( \hat{p} \models (M[\hat{H}] \bowtie 2^\omega) < \beta + 1 \) compatible with \( p \) and \( \forall q \in P^\gamma \) if \( |q|(M[\hat{H}] \bowtie 2^\omega) < \beta \), then \( (\hat{p}, q \) compatible \( \rightarrow p, q \) compatible).

Proof.
This is proved by induction on \( \gamma \). For \( \gamma \) limit it is easy, also for \( \gamma + 1 \) in which Case 1 occurs the proof is the same as Lemma 25. So we only have to do \( \gamma + 1 \) in Case 2.
\( p \in P^\gamma \ast P_{B_1}(Y_\gamma - F, X_{\beta_1} \cup F). \) Extend \( p(\gamma) \) if necessary so that \( \forall < s, x > \in p(\gamma) \forall i < \omega \text{ if } |s| = \lambda \text{ infinite limit } |s^i| \leq \beta + 1 < \lambda \text{ then } \exists j < \omega < s^i \cap j, x > \in p(\gamma). \)

Let \( \hat{p}(\gamma) = \{ < s, x > \in p(\gamma) : |s| < \beta + 1 \text{ or } x \neq z \}. \)

If \( \hat{p} = < \hat{p}_Y, \hat{p}(\gamma) > \) were a condition then just as in Lemma 8, \( \hat{p} \) would have the required properties. To be a condition we need to know that whenever \( < < n, x > \in \hat{p}(\gamma) \)
\( \hat{p}_Y \models "x \notin (Y_\gamma - F)". \)

Note that none of these \( x \)'s are equal to \( z \) because \( z \in X_{\alpha + 1} \) so \( < < n, z > \in p(\gamma) \rightarrow |< n, >| = \alpha \geq \beta + 1 \) so \( < < n, z > \notin \hat{p}(\gamma). \) Let \( G \) be \( P^\gamma \)-generic containing \( p_\gamma \), and \( \hat{p}_Y \). By Lemma 31 \( \exists q \in P^\gamma_H \cap G. \) (So \( |q|(M[H] \cap 2^\omega) = 0 \)) \( q \models "x \notin Y_\gamma - F" \forall x \forall n < < n, x > \in \hat{p}(\gamma). \) By Lemma 23, \( \exists p_0 \geq q, \hat{p}_Y \text{ so that } |p_0|(M[H] \cap 2^\omega) < \beta + 1. \) So \( < p_0, \hat{p}(\gamma) > \) works.

Immediate from Lemma 32 we get that: If \( J \) is any \( \Sigma^0_{\alpha + 1} \) predicate with parameters \( (H = X - \{z\}, z \in X_{\alpha + 1}, \) and \( \tau \) is in the forcing language of \( P_H \) then \( \forall p \in P \text{ if } p \models "z \in J" \text{ then } \exists q \in P \text{ if } |q|(M[H] \cap 2^\omega) < \beta, q \text{ and } p \text{ are compatible, and } q \models "z \in J". \) So we get our result \( \text{ord}(X_{\alpha + 1}) = \alpha + 1 \text{ in } M_0[G_{\omega_2}]. \)

Remark: Assuming large amounts of the axiom of determinacy and therefore getting more absoluteness in inner models
(see [7]) it is easy to produce an inner model $N$ such that $N \models \text{"For every } \alpha < \omega_1 \text{ there exist } X \subseteq 2^\omega \text{ such that } \text{ord}(X) = \alpha \text{ and for every } n < \omega \text{ and } A \Pi^1_n A \subseteq X \text{ is Borel in } X"$. Similar improvements for Theorem 43 are possible.
§4. The $\sigma$-algebra generated by the abstract rectangles

For any cardinal $\lambda$ let $R^\lambda = \{A \times B: A, B \subseteq \lambda\}$. If $R^\omega_{\omega_1}$ (the $\sigma$-algebra generated by $R^\lambda$) is the set of all subsets of $\lambda \times \lambda$, then $\lambda \leq |2^\omega|$ (see [9], [21]).

Theorem 33. If $\alpha_0 < \omega_1$ and there is an $X \subseteq \omega^\omega$ such that $|X| = \kappa \geq \omega$ and every subset of $X$ of cardinality less than $\kappa$ is $\prod_{\alpha_0}^\theta$ in $X$, then $R^\kappa_{\alpha_0} = P(\kappa \times \kappa)$. The same is true if every subset of $X$ of cardinality less than $\kappa$ is $\sum_{\alpha_0}^\theta$ in $X$.

Proof. Consider $A \subseteq \kappa \times \kappa$ and suppose $(\alpha, \beta) \in A$ implies $\alpha \leq \beta$. It is enough to show such sets are in $R^\kappa_{\alpha_0}$, since every subset of $\kappa \times \kappa$ can be written as the union of a set above the diagonal and a set below the diagonal. Let $T$ be a normal $\alpha_0$ tree and $T^* = \{s \in T: |s|_T = 0\}$. For any $y: T^* \to \omega^\omega$ define $G^y_s$ as follows. If $s \in T^*$, then $G^y_s = [y(s)]$, otherwise $G^y_s = \bigcap \{\omega^\omega - G^y_{y_n}: n < \omega\}$. Let $X = \{x_\alpha: \alpha < \kappa\}$ and for each $\beta < \kappa$ choose $\beta$ so that for all $\alpha$ ($(\alpha, \beta) \in A$ iff $x_\alpha \in G^\phi_y_{\beta}$). For $s \in T$ define $B_s \subseteq \kappa \times \kappa$ as follows. If $s \in T^*$, then $B_s = \bigcup \{\{\alpha: t \subseteq x_\alpha\} \times \{\beta: y_{\beta}(s) = t\}: t \in \omega^\omega\}$, otherwise $B_s = \bigcap \{\kappa \times \kappa - B_{s\wedge n}: n < \omega\}$. Clearly $B_\phi = A$ and
B_φ is "Π_α^0" in R^K, and so every subset of κ × κ is "Π_α^0" in R^K. Note that (κ × κ) - (A × B) = ((κ - A) × κ) ∪ (κ × (κ - B)) and thus if α_0 is even (odd), then R^K_{α_0} is the class of sets "Π_α^0" ("Σ_α^0") in R^K. By passing to complements if necessary we have that R^K_{α_0} = P(κ × κ). The second sentence of the theorem is proved similarly.

Corollary. (Kunen [9]; Rao [21]) If there is an X ⊆ 2^ω such that |X| = ω₁ then R^{ω₁} = P(ω₁ × ω₁).

The converse of this corollary is also true. Suppose R ⊆ P(ω₁) is a countable field of sets and 
{(α,β): α < β < ω₁} ∈ {A × B: A,B ∈ R}^ω₁. Since this set is antisymmetric we conclude that the map given in Proposition 2 is a 1-1 embedding of ω₁ into 2^ω.

Corollary. (Kunen [9]; Silver) (MA) if κ = |2^ω| then R^K_2 = P(κ × κ).

Proof.
If X is I-Luzin where I is the ideal of meager sets, then every subset of X of smaller cardinality is Σ_2^0 in X (see proof of Theorem 17).
For any $\alpha \leq \omega_1$, $X \subseteq \omega^\omega$ is a $Q_\alpha$ set iff $\text{ord}(X) = \alpha$ and every subset of $X$ is Borel in $X$.

**Theorem 34.** If $M$ is countable transitive model of ZFC, $1 \leq \alpha_0 < \omega^M_1$, and $X = M \cap \omega$, then there is a Cohen extension $M[G]$ such that $M[G] \models "X$ is a $Q_{\alpha_0+1}$ set".

**Remark:** This shows that the Baire order of the constructible reals can be any countable successor ordinal greater than one. In fact the argument shows that in $M[G]$ for any uncountable $Y \subseteq 2^\omega$ with $Y \in M$ $Y$ is a $Q_{\alpha_0+1}$ set. Thus, for example, if $M$ models $V = L$, then in $M[G]$ there are $\Pi^1_1$ $Q_{\alpha_0+1}$ sets. In Theorem 55 we show that it is consistent with ZFC that for every $\alpha < \omega_1$ there is a $Q_\alpha$ set (in that model the continuum is $\aleph_{\omega_1+1}$).

**The proof of Theorem 34:**

$M[G]$ is gotten by iterated $\Pi^0_\alpha$-forcing. Let $\kappa = |2^{2^\omega}|$. Suppose we are given $\mathbb{P}^\alpha$ for some $\alpha < \kappa$ and $Y_\alpha$ a term in the forcing language of $\mathbb{P}^\alpha$ for a subset of $X$ ($\phi \models "Y_\alpha \subseteq X"$), then let $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha \ast \mathbb{P}_{\alpha+1}(Y_\alpha, X)$. At limit ordinals take direct limits. $\mathbb{P}^\kappa$ may be viewed as a sub-lower lattice of $\bigcap_{\kappa} \mathbb{P}_{\alpha+1}(\phi, X)$. We may assume that
for every set \( B \subseteq X \) in \( M[G] \) (\( G \mathbb{P}^K \)-generic over \( M \)) there exists \( \alpha \) such that \( Y_\alpha = B \). This is because \( \mathbb{P}^K \) has c.c.c. It follows from Corollary 6 that \( M[G] \vDash \text{"ord}(X) \leq \alpha_0 + 1 \) and every subset of \( X \) is Borel in \( X \).

We assume \( \mathbb{P}^0 = \mathbb{P}_{\alpha_0+1}(\emptyset, X) \). Let \( G(0) \) be one of the \( \Pi^0_{\omega_1} \) set determined by \( G \cap \mathbb{P}^0 \). We want to show that \( M[G] \vDash \text{"For every } K \text{ in } \Sigma^0_{\alpha_0}, K \cap X \notin G(0) \cap X \". To this end we make the following definition: For \( H \subseteq \omega^\omega \), \( |p|(H) = \max\{|s|: \text{there exists } x \notin H \text{ (s,x) } \in p(\alpha) \text{ for some } \alpha < \kappa \} \). Let \( \text{supp}(p) = \{ \alpha < \kappa: p(\alpha) \neq \emptyset \} \). Given \( \tau \) a term in the forcing language of \( \mathbb{P}^K \) denoting a subset of \( \omega \), we can find \( H \) included in \( \omega^\omega \) and \( K \) included in \( \kappa \) with the following properties:

(a) \( H \) and \( K \) are countable.
(b) for each \( n \in \omega \) \( \{ p \in \mathbb{P}^K: \text{supp}(p) \subseteq K, |p|(H) = 0 \} \), decides "\( n \in \tau \".
(c) \( \forall x \in H \forall \alpha \in K \{ p \in \mathbb{P}^K: \text{supp}(p) \subseteq K, |p|(H) = 0 \} \) decides "\( x \in Y_\alpha \".

\( H \) and \( K \) can be found by repeatedly using the c.c.c. of \( \mathbb{P}^K \).

**Lemma 35.** If \( H \) and \( K \) have property (c) then for any \( p \in \mathbb{P}^K \) and \( \beta \) with \( 1 \leq \beta < \alpha_0 \), there exists \( \hat{p} \in \mathbb{P}^K \) compatible with \( p \), \( |\hat{p}|(H) < \beta + 1 \), \( \text{supp}(\hat{p}) \subseteq K \), and for
any \( q \in \mathbb{P}^K \) if \( |q|(H) < \beta \) and \( \text{supp}(q) \subseteq K \) then

[if \( \hat{p} \) and \( q \) are compatible, then \( p \) and \( q \) are compatible].

Proof.

The proof of this is like Lemma 8.

Let \( G \) be \( P^K \)-generic over \( M \) with \( p \in G \). Choose \( \Gamma \subseteq G \) finite with the properties:

1. \( \forall q \in \Gamma (|q|(H) = 0 \) and \( \text{supp}(q) \subseteq K \).

2. If \( ((n),x) \in p(\alpha) \) for some \( n < \omega, \alpha \in K \), and \( x \in H \) (so \( p \models \alpha \models \neg x \notin Y_\alpha \)), then there is \( q \in \Gamma \cap \mathbb{P}^\alpha \) such that \( q \models \neg x \notin Y_\alpha \).

3. If \( (s,x) \in p(\alpha), \alpha \in K, \) and \( |s| = \lambda \) is an infinite limit ordinal, and \( |s^\omega| \leq \beta + 1 < \lambda \) then there is a \( j \in \omega \) such that \( \{(s^\omega j,x)\} \in \Gamma \).

Now let \( \hat{p} \in \mathbb{P}^K \) be defined by

\[
\hat{p}(\alpha) = \bigcup \{ r(\alpha) : r \in \Gamma \} \cup \{(s,x) \in p(\alpha) : |s| < \beta + 1 \text{ or } x \in H\}
\]

when \( \alpha \in K \) and \( \hat{p}(\alpha) = \emptyset \) for \( \alpha \notin K \).

Note if \( ((n),x) \in \hat{p}(\alpha) \) then \( x \in H \) since \( |(n)| = \alpha_0 \geq \beta + 1 \). By choice of \( \Gamma \) \( \hat{p} \) is a condition and also \( |\hat{p}|(H) < \beta + 1 \) and is compatible with
A p since p, p ε G. It is easily checked as in Lemma 8 that p has the required property.

**Lemma 36.** Let H and K have properties (b) and (c) for τ. Let B(v) be a \( \Sigma_\beta^0 \) (1 ≤ \( \beta \) ≤ \( \alpha_0 \)) predicate with parameters from M and p ε \( \mathcal{P}^\alpha \) such that p\( \vDash "B(\tau)" \). Then there exists q ε \( \mathcal{P}^\alpha \) compatible with p, |q|(H) < \( \beta \), q\( \vDash "B(\tau)" \), and \( \text{supp}(q) \subseteq K \).

**Proof.**

The proof is by induction on \( \beta \).

\( \beta = 1: \)

p\( \vDash "\exists n R(n, \tau\vec{n}, x\vec{n})" \), x ε M, and R is primitive recursive. Let G be \( \mathcal{P} \)-generic over M with p ε G. There exist n ε \( \omega \) and s ε \( 2^n \) such that M[G] \( \vDash "R(n, \tau\vec{n}, x\vec{n}) \) and \( \tau\vec{n} = s" \). By property (b) there exists q ε G such that q\( \vDash "\tau\vec{n} = s" \), \( \text{supp}(q) \subseteq K \), and \( |q|(H) = 0 \). q does it.

\( \beta \) limit:

p\( \vDash "\exists n B_n(\tau)" \), B_n \( \in \Sigma_\beta^0 \), \( \beta_n < \beta \). Choose r \( \geq p \) such that r\( \vDash "B_n(\tau)" \) for some n. By induction there exist q such that q\( \vDash "B_n(\tau)" \), q is compatible with r (and hence with p), and \( |q|(H) < \beta \), \( \text{supp}(q) \subseteq K \). q does it.

\( \beta + 1: \)

If p\( \vDash "\exists n B_n(\tau)" \) we could extend p to force B_n(\( \tau \)) for some particular n. So we may as well assume p\( \vDash "B(\tau)" \) where B(v) is \( \Sigma_\beta^0 \) with parameter in M. Since 1 \( \leq \beta \) \( \leq \alpha_0 \) by
Lemma 35 there is $\hat{p}$ compatible with $p$, $|\hat{p}|(H) < \beta + 1$, etc. Then $\hat{p} \models "B(\tau)"$ because otherwise there is $p_0 \geq \hat{p}$ such that $p_0 \models "B(\tau)"$, and so by induction there is $q$ compatible with $p_0$ (hence with $\hat{p}$) $|q|(H) < \beta$, $\text{supp}(q) \subseteq K$, and $q \models "B(\tau)"$. By our assumption on $\hat{p}$, since $\hat{p}$ and $q$ are compatible, $p$ and $q$ are compatible, but $p \models "B(\tau)"$. 

We now use Lemma 36 to show that for any $G \mathcal{P}^K$-generic over $M$, $M[G] \models "\text{For every } L \text{ a } \Sigma^0_{\alpha_0} \text{ set } (L \cap X \not= G_{(o)} \cap X)"$ where $G_{(o)}$ is one of the $\Pi^0_{\alpha_0}$ sets determined by $G \cap \mathcal{P}_{\alpha_0+1}(\phi,X)$. Suppose not; then let $\tau$ be a term in forcing language of $\mathcal{P}, L$ a $\Sigma^0_{\alpha_0}$ set with parameter $\tau$, and $p \in G$ such that $p \models "\text{for every } x \in X x \in L \text{ iff } x \in G_{(o)}"$. Choose $H$ and $K$ with properties (a), (b), and (c) with respect to $\tau$ and also so that $\text{supp}(p) \subseteq K$ and $|p|(H) = 0$. Since $H$ is countable there exists $x \in X - H$. Let $r = p \cup \{(0,((0),x))\}$ (so $r \models "x \in G_{(o)}"$). Since $r \models "x \in L"$, by Lemma 36 there exists $q$ compatible with $r$, $|q|(H) < \alpha_0$, and $q \models "x \in L"$. Since $|q|(H) < \alpha_0$, $((0),x) \not\in q(0)$. Let $\hat{q}$ be defined by:
\( p(\alpha) \cup q(\alpha) \)

if \( \alpha > 0 \).

\( \hat{q}(\alpha) = \)

\( p(0) \cup q(0) \cup \{((0,m),x)\} \)

otherwise (\( m \) sufficiently large so that

\( \hat{q}(0) \) is a condition).

\[ \hat{q} \models "x \in L \text{ and } x \notin G(0) \text{ and } (x \in L \iff x \in G(0))". \]

This a contradiction and concludes the proof of Theorem 34.

Theorem 37. For any \( \alpha_0 \) a successor ordinal such that

\( 2 \leq \alpha_0 < \omega_1 \), it is relatively consistent with ZFC that

\[ |2^\omega| = \omega_2 \]

and \( \alpha_0 \) is the least ordinal such that

\[ R_{\alpha_0}^{\omega_2} = P(\omega_2 \times \omega_2) \].

Remark: In Theorem 52 we remove the restriction that \( \alpha_0 \)

is a successor (but the continuum in that model is

\( \aleph_{\omega+1} \)). In [1] it is shown that \( \alpha_0 \) cannot be \( \omega_1 \).

Proof.

Let \( M \) be a countable transitive model of "ZFC +

\[ |2^\omega| = |2^{\omega_1}| = \omega_2". \]

Let \( X = \omega^\omega \cap M \) and define \( P_\alpha \)

for \( \alpha \leq \omega_2 \) so that \( P_{\alpha+1} = P_\alpha \cdot P_{\alpha_0}(Y_\alpha, X) \) where \( Y_\alpha \)

is a \( P_\alpha \) term for a subset of \( X \), and at limits take the direct

limit. Dovetail so that in \( M[G_{\omega_2}] \) for every \( Y \subseteq X \) such

that \( |Y| \leq \omega_1 \) there are \( \omega_2 \) many \( \alpha < \omega_2 \) such that

\( Y_\alpha = Y \). By Theorem 33 \( R_{\alpha_0}^{\omega_2} = P(\omega_2 \times \omega_2) \).
Now comes the difficulty: we must show some subset of \( \omega_2 \times \omega_2 \) is not in \( R^{\omega_2}_{\alpha_0} \). For the remainder of the proof let \((A_s: s \in \omega^{<\omega}) \) and \((B_s: s \in \omega^{<\omega}) \) be fixed terms in the forcing language of \( \mathbb{P}^{\omega_2} \) such that for every \( s \in \omega^{<\omega} \) \( \vdash "A_s \subseteq X \text{ and } B_s \subseteq \omega_2" \). For \( p \in \mathbb{P}^{\omega_2} \) define \( \text{supp}(p) = \{\alpha < \omega_2: p(\alpha) \neq \phi\} \) and \( \text{trace}(p) = \{x \in X: \exists \alpha \exists t(t,x) \in p(\alpha)\} \). By using the c.c.c. of \( \mathbb{P}^{\omega_2} \) choose for each \( x \in X \) countable sets \( I_x \subseteq X \) and \( J_x \subseteq \omega_2 \) so that:

1. For each \( s \in \omega^{<\omega} \) \( \{p \in \mathbb{P}^{\omega_2}: \text{trace}(p) \subseteq I_x \text{ and } \text{supp}(p) \subseteq J_x\} \) decides "\( x \in A_s \)".
2. For each \( y \in I_x \) and \( \alpha \in J_x \) \( \{p \in \mathbb{P}^{\omega_2}: \text{trace}(p) \subseteq I_x \text{ and } \text{supp}(p) \subseteq J_x\} \) decides "\( y \in Y_\alpha \)"

Similarly for \( \alpha < \omega_2 \) we can pick countable sets \( I_\alpha \subseteq X \) and \( J_\alpha \subseteq \omega_2 \) having properties (1) and (2) with \( \alpha, B_s, I_\alpha, I_\alpha \) in place of \( x, A_s, I_x, I_x \).

For \( x \in X \) and \( \alpha < \omega_2 \) let

\[
L(x,\alpha) = (I_x \times J_x) \cup (I_\alpha \times J_\alpha)
\]

and define for \( p \in \mathbb{P}^{\omega_2} \),

\[
|p|(x,\alpha) = \max\{|s|_T: (s,u) \in p(\gamma) \text{ and } (u,\gamma) \notin L(x,\alpha)\}.
\]

**Lemma 38.** Fix \( x \in X \) and \( \alpha < \omega_2 \) and let \( |p| = |p|(x,\alpha) \).

For any \( \beta \geq 1 \) and \( p \in \mathbb{P}^{\omega_2} \) there is a \( \hat{p} \in \mathbb{P}^{\omega_2} \) with

\[
|\hat{p}| < \beta + 1, \hat{p} \text{ compatible with } p, \text{ and for any } q \in \mathbb{P}^{\omega_2}
\]

if \( |q| < \beta \) and \( \hat{p} \) and \( q \) are compatible, then \( p \) and
q are compatible.

Proof.
The proof of this is like that of Lemma 35. Let \( p_0 \geq p \) so that if \( (s,x) \in p(\gamma) \) with \(|s| = \lambda \) a limit ordinal greater than \( \beta \) and \(|s^i| \leq \beta + 1\), then there is \( j < \omega \) so that \( (s^i,j,x) \in p_0(\gamma) \). Let \( G \) be \( \mathbb{P}^{\omega_2} \)-generic with \( p_0 \in G \). Choose \( \Gamma \subseteq G \) finite so that if

\[
((n),u) \in p_0(\gamma) \quad (\text{so } p_0 \not\Vdash \ulcorner u \notin Y_\gamma \urcorner)
\]

\((u,\gamma) \in L(x,\alpha)\), then there is a \( q \in \Gamma \) such that

\( q \Vdash \ulcorner u \notin Y_\gamma \urcorner \). Define \( \hat{p} \) by

\[
\hat{p}(\gamma) = \bigcup \{q(\gamma) : q \in \Gamma\} \cup \{(s,u) \in p_0(\gamma) : |s| < \beta + 1 \text{ or } (u,\gamma) \in L(x,\alpha)\}.
\]

For any well founded tree \( \hat{T} \) define \( C_s(\hat{T}) \) for \( s \in \hat{T} \) as follows. If \(|s|_{\hat{T}} = 0\) then \( C_s(\hat{T}) = A_s \times B_s \), otherwise \( C_s(\hat{T}) = \bigcup \{(X \times \omega_2) - C_{s^i}(\hat{T}) : i < \omega\} \).

Lemma 39. If \( x \in X, \alpha \in \omega_2, \hat{T} \in M \) is a well founded tree, \( s \in \hat{T} \) with \(|s|_{\hat{T}} = \beta \) where \( 1 \leq \beta \leq \alpha_0 - 1 \), and \( p \in \mathbb{P}^{\omega_2} \) such that \( p \Vdash \ulcorner (x,\alpha) \notin C_s(T) \urcorner \), then there exist \( q \) compatible with \( p \), \(|q|(x,\alpha) < \beta \), and

\( q \Vdash \ulcorner (x,\alpha) \notin C_s(T) \urcorner \).

Proof.
The proof is by induction on \( \beta \).
Case 1. $\beta = 1$:

Suppose $p \vdash "(x,\alpha) \in \bigcup_{i \in \omega} (A_{s^\alpha_i} \times B_{s^\alpha_i})"$. So there exists $i_0 \in \omega$ and $\hat{p}$ and $\hat{q}$ elements of $\Pi^\omega$ so that $(p \cup \hat{p} \cup \hat{q}) \in \Pi^\omega$ and using (1) above

$(t,u) \in \hat{p}(\gamma) \iff (u,\gamma) \in I_x \times J_x$ and

$(t,u) \in \hat{q}(\gamma) \iff (u,\gamma) \in I_\alpha \times J_\alpha$ and

$\hat{p} \vdash "x \in A_{s^\alpha_{i_0}}", \hat{q} \vdash "y \in B_{s^\alpha_{i_0}}"$. So $\hat{p} \cup \hat{q} = q$ does the job.

Case 2. $\beta$ a limit ordinal:

Suppose $p \vdash "(x,\alpha) \in \bigcup_{i \in \omega} C_{s^\alpha_i}(\hat{T})$ where $|s|^\hat{T} = \beta$. Find $q \supset p$ and $i_0 \in \omega$ such that $q \vdash "(x,y) \in C_{s^\alpha_{i_0}}(\hat{T})"$. Let

$T_0 = \{t \in \hat{T}: s^\alpha_{i_0} \leq t \text{ or } t \in s^\alpha_{i_0}\}$. Then

$|s|_{T_0} = |s^\alpha_i|^\hat{T} + 1 < \beta$ and $C_s(T_0) = (X \times \omega_2) - C_{s^\alpha_{i_0}}(T)$, hence $q \vdash "(x,\alpha) \notin C_s(T_0)"$ where $|s|_{T_0} < \beta$; so by induction hypothesis there exists $r$ compatible with $q$ (and hence with $p$), $|r|(x,\alpha) < \beta$, and

$r \vdash "(x,\alpha) \in C_{s^\alpha_{i_0}}(T)"$. $r$ does the trick.

Case 3. $\beta + 1$:

Since $\beta + 1 < \alpha_0$, let $q$ be as from Lemma 38. $\square$

Define $D \subseteq X \times \omega_2$ by $D = \{(x,\alpha): x \in G^\alpha_{(0)} \}$ where $G^\alpha_{(0)}$ is one of the $\Pi^0_{\alpha_0 - 1}$ sets created on the $\alpha$th step. $D$ is $\Pi^0_{\alpha_0 - 1}$ in the rectangles on $X \times \omega_2$. We want to show it is not $\Sigma^0_{\alpha_0 - 1}$ in the rectangles on $X \times \omega_2$ in $M[G_{\omega_2}]$. 
Define: \((x, \alpha)\) is free (with respect to 
\((A_s: s \in \omega^{<\omega}, B_s: s \in \omega^{<\omega})\) iff \([x \not\in I_{\alpha} \text{ and } \alpha \not\in J_x]\).

**Lemma 40.** If \(T \subseteq \omega^{<\omega}\) is well founded and \(T \in M, s \in T\) with \(|s|_T \leq \alpha_0 - 1, (x, \alpha)\) is free, and \(Y_\alpha = \emptyset\); then for every \(p \in \mathbb{P}^{\omega^2}\) such that \(|p|(x, \alpha) = 0\) it is not the case that \(p \models "(x, \alpha) \in D \text{ iff } (x, \alpha) \not\in C_s(T)"\).

**Proof.**

Let \(\hat{p} \geq p\) by defining \(\hat{p}(\gamma) = p(\gamma)\) for \(\gamma \neq \alpha\) and \(\hat{p}(\alpha) = p(\alpha) \cup \{(0), x\}\). Then \(\hat{p} \models "(x, \alpha) \in D"\) so by Lemma 39 there exists \(q\) compatible with \(\hat{p}\), \(|q|(x, \alpha) < \alpha_0\), and \(q \models "(x, \alpha) \not\in C_s(T)"\). But \((x, \alpha)\) free implies that \((x, \alpha) \not\in L(x, \alpha)\) so \(q\) does not say "\(x \in \bigcap_{\gamma \in (0)} Y_{\alpha}\)". Thus for a sufficiently large \(m < \omega\) \(r\) defined by \(r(\gamma) = p(\gamma) \cup q(\gamma)\) for \(\gamma \neq \alpha\) and \(r(\alpha) = p(\alpha) \cup q(\alpha) \cup \{(0, m), x\}\) is a member of \(\mathbb{P}^{\omega^2}\).

But \(r \models "(x, \alpha) \not\in D \text{ and } (x, \alpha) \not\in C_s(T)"\), a contradiction since \(r\) extends \(p\).

Since the terms \((A_s: s \in \omega^{<\omega})\) and \((B_s: s \in \omega^{<\omega})\) were arbitrary to start with it will complete the proof of the theorem to find lots of \((x, \alpha)\) free.

The next lemma generalizes Kunen [9], p. 74.
Lemma 41. Given $|I_\alpha| < \kappa$ for $\alpha < \kappa^+$, there exists $G \subseteq \kappa^+$ with $|G| = \kappa^+$ and there is $S$ with $|S| \leq \kappa$ so that for any $\alpha, \beta \in G$ if $\alpha \neq \beta$ then $I_\alpha \cap I_\beta \subseteq S$.

Proof.
We can assume $I_\alpha \subseteq \kappa^+$.
Define $u_\alpha, z_\alpha < \kappa^+$ for $\alpha < \kappa^+$ nondecreasing so that:

1. $u_\lambda = \sup\{u_\alpha : \alpha < \lambda\}$ for $\lambda$ limit;
2. $z_\alpha$'s are strictly increasing;
3. for $\alpha$ a successor and for distinct $\beta, \gamma < \alpha$ $I_{z_\beta} \cap I_{z_\gamma} \subseteq u_\alpha$;
4. if $u_{\alpha+1} > u_\alpha$ then for any $z > z_\alpha$ $u_\alpha \not\in I_Z \cap \bigcup\{I_{z_\beta} : \beta < \alpha\}$ and $\bigcup\{I_{z_\beta} : \beta < \alpha\} \subseteq u_{\alpha+1}$.

Let $G = \{z_\alpha : \alpha < \kappa^+\}$ and $S = \sup\{u_\alpha : \alpha < \kappa^+\}$.
To see that $S < \kappa^+$ note that for any $\alpha < \kappa^+$ $|\{\beta : u_{\beta+1} > u_\beta$ and $\beta < \alpha\}| < \kappa$. This is because $I_{z_\alpha} \cap (u_{\beta+1} - u_\beta) \neq \emptyset$ for all $\beta < \alpha$ such that $u_{\beta+1} > u_\beta$. 

Lemma 42. There exists $\Sigma_0 \subseteq X$ and $\Sigma_1 \subseteq \omega_2$ with $|\Sigma_0| = |\Sigma_1| = \omega_2$, for every $\alpha \in \Sigma_1$, $Y_\alpha = \emptyset$, and for every $(x, \alpha) \in \Sigma_0 \times \Sigma_1$ $(x, \alpha)$ is free.

Proof.
By Lemma 41 there exists $\hat{\Sigma}_0 \subseteq X$ and $S \subseteq \omega_2$ with $|\hat{\Sigma}_0| = \omega_2$ and $|S| < \omega_2$ so that for every distinct
x, y ∈ Σ₀, Jₓ ∩ Jᵧ ⊂ S. Since \{Jₓ ∩ S : x ∈ Σ₀\} is a disjoint family, we can cut down \(\Sigma_0\) (maintaining \(|\Sigma_0| = \omega_2\)) and find \(\Sigma_1 ⊂ \omega_2\) so that \(|\Sigma_1| = \omega_2\), for every \(α ∈ \Sigma_1\), \(Y_α = \emptyset\), and for every \(x ∈ \Sigma_0\), \(J_x \cap \Sigma_1 = \emptyset\).

Applying Lemma 41 again find \(Σ₁ ⊆ Σ₁\) with \(|Σ₁| = ω₂\) and \(T ⊆ X\) with \(|T| < ω₂\) so that for every distinct \(α, β ∈ Σ₁\), \(I_α \cap I_β ⊂ T\). Since \(\{I_α - T : α ∈ Σ₁\}\) are disjoint by cutting down \(Σ₁\) (maintaining \(|Σ₁| = ω₂\)) we can assume \(Σ₀\) defined to be equal to \(Σ₀ - (T ∪ ∪ \{I_α : α ∈ Σ₁\})\) has cardinality \(ω₂\). \(Σ₀\) and \(Σ₁\) do the job.

Lemma 42 finishes the proof of Theorem 37.

Remark: There is nothing special about \(ω₂\) in the above theorem; we could have replaced it by any larger cardinal \(κ\) with \(κ^κ = κ\).
Now we turn to a slightly different problem. For $X$ a topological space, a set $A \subseteq X^n$ is projective if and only if it is in the smallest class containing the Borel sets (in the product topology on $X^m$ for any $m \in \omega$) and closed under complementation and projection ($B \subseteq X^m$ is the projection of $C \subseteq X^{m+1}$ iff $(\forall \bar{y} \in B \iff \exists x \in X \ x\bar{y} \in C)$).

**Theorem 43.** If $M$ is a countable transitive model of ZFC then there exists $N$ a c.c.c. Cohen extension of $M$ such that if $M^\omega = X$ then $N \models \text{"Every projective set in } X \text{ is Borel and the Borel hierarchy of } X \text{ has } \omega_1\text{ distinct levels (ord}(X) = \omega_1\text{"").}$$

This shows the relative consistency of an affirmative answer to a question of Ulam ([31], p. 10). Note that since $X \times X$ is homeomorphic to $X$ (take any recursive coding function), if for every $B \subseteq X \times X$ Borel \{x: \exists y(x,y) \in B\} is Borel in $X$, then every projective set in $X$ is Borel in $X$.

**Proof.**
The proof is slightly simpler if we assume that CH holds in $M$. We give the proof in that case and then later indicate the necessary modifications. In any case $|2^\omega|^M = |2^\omega|^N$.

Construct a sequence $M = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{\omega_1} = N$, by iterated forcing so that $M_{\alpha+1}$ is obtained from $M_\alpha$ by $\Gamma_\alpha$-forcing. On the $\alpha^{th}$ stage we are presented with a
term $\tau_\alpha$ in the forcing language of $P^\alpha$ denoting a real. Then letting $Y_\alpha$ be the projective set (over $X$) determined by $\tau_\alpha$ we let $P^{\alpha+1} = P^\alpha \ast P_{\alpha+1}(Y_\alpha, X)$. What is being done is that at stage $\alpha$ we make $Y_\alpha$ a $\Sigma^{\omega_1}_{\alpha+1}$ set intersected with $X$. The reason this will work is that after the $\alpha^{th}$ stage our forcing will not interfere with the Borel hierarchy on $X$ up to the $\alpha^{th}$ level. Since this is c.c.c. forcing we can imagine that each $X$-projective set in $N$ is eventually caught by some $\tau_\alpha$ for $\alpha < \omega_1$. So it is clear that $N \models $ "Every $X$-projective set is Borel in $X$", for any $N = M[G]$, where $G$ is $P^{\omega_2}$-generic over $M$. Define for $H \subseteq X$ and $p \in P$, $|p|(H) = \max \{|s|_{T_{\alpha+1}} : \text{there exist } \alpha < \omega_1 \text{ and } x \notin H (s,x) \in p(\alpha)\}$. Given $\tau$ a term in the forcing language of $P^\gamma$ denoting a subset of $\omega$ ($\gamma < \omega_1$), there exists $H \subseteq X$ such that:

(a) $H$ is countable.

(b) $\forall n \in \omega$

\[\{p \in P^\gamma : |p|(H) = 0\} \text{ decides } "n \in \tau".\]

(c) $\forall \beta < \gamma$ and $x \in H$

\[\{p \in P^\gamma : |p|(H) = 0\} \text{ decides } "x \in Y_\beta".\]

**Lemma 44.** (write $|p| = |p|(H)$).

"Exactly statement of Lemma 38" for $P^\gamma$.

**Proof.**

Extend $p \leq p^*$ as before. Let $G$ be $P^\gamma$-generic with
p_0 \in G. Choose \Gamma \subseteq G finite so that:

(1) \ q \in \Gamma \rightarrow |q|(H) = 0;
(2) if \symb{n, x} \in p_0(\alpha) \ (so \ p_\alpha \models "x \notin \gamma") \ then

\exists q \in \Gamma \cap D^\alpha \ \text{such that} \ q \models "x \notin \gamma".

Define \hat{p}(\alpha) = \bigcup\{r(\alpha) : r \in \Gamma \} \cup \{s, x \in p_0(\alpha) : |s|_{\alpha} < \beta + 1 \text{ or } x \in H\}. \hat{p} \text{ is a condition because if}

\symb{n, x} \in p(\alpha) \text{ and } |\symb{n}|_{\alpha+1} < \beta + 1, \text{ then } \hat{p}_\alpha \geq p_\alpha

\text{(so } \hat{p}_\alpha \models "x \notin \gamma" \text{ as required).}

The \ r \in \Gamma \text{ take care of such requirements about } x \in H.

The rest of the proof is the same.

Lemma 45. If \tau, H, \gamma \text{ are as above, } B(v) \text{ is a } \Sigma^0_\beta \text{ predicate for some } \beta \geq 1 \text{ with parameter from } M, \text{ and } p \in P^\gamma \text{ such that } p \models "B(\tau)", \text{ then there is a } q \in P^\gamma \text{ compatible with } p, |q|(H) < \beta \text{ and } q \models "B(\tau)".

Proof. The proof is the same as before.

We can assume that for unboundedly many \alpha < \omega_1 \ Y_\alpha = \emptyset.
Let \ G_\alpha (G_\alpha) \text{ be one of the } \Pi^0_\alpha \text{ sets determined by}

\ G \land P_{\alpha+1}(\emptyset, X) \text{ where } Y_\alpha = \emptyset.

Claim: M[G] \models "for any } L \in \Sigma^0_\alpha \ (L \land X \neq G_\alpha \land X)".
Proof.

Otherwise let \( \tau \) be a term for a real in the forcing language \( P^Y \) for some \( \gamma < \omega_1 \) such that for some \( L \in \Sigma^0_\alpha \) set with parameter \( \tau \) and some \( p \in P^Y \)
\( p \models "L \cap X = G_\alpha \cap X". \) Choose \( H \) with properties (a), (b), and (c) with respect to \( \tau \), and also \( |p|(H) = 0. \) Let \( x \in X - H. \) Define \( r(\alpha) = p(\alpha) \cup \{(0,x)\} \) and for \( \beta \neq \alpha \) \( r(\beta) = p(\beta). \) Note that \( r \models "x \in G_\alpha" \) hence
\( r \models "x \in L". \) By Lemma 45 there exists \( q \in P^Y \) compatible with \( r, |q|(H) < \beta, \) and \( q \models "x \in L". \) Since \( x \notin H \) we know \( ((0),x) \notin q(\alpha). \) Define \( \hat{q} \in P^{\omega_1} \) by
\( \hat{q}(\beta) = p(\beta) \cup q(\beta) \) for \( \beta \neq \alpha \) and \( \hat{q}(\alpha) = p(\alpha) \cup q(\alpha) \cup \{(0,n,x)\} \) where \( n \) is picked sufficiently large so \( \hat{q}(\alpha) \) is a condition. But then \( \hat{q} \models "x \in L \) and \( x \notin G_\alpha \) and \( (x \in L \iff x \in G_\alpha)" \) and this is a contradiction.
This concludes the proof of Theorem 43. 

When the continuum hypothesis does not hold in \( M \) the construction of \( N \) still has \( \omega_1 \) steps but at each step we must take care of all reals in the ground model. That is \( P^{\omega_1} = P^{\omega_1} * Q_\alpha \) where \( Q_\alpha \) is a term denoting \( \Sigma \{ P_{\alpha+1}(H_x, X) : x \in \omega^\omega \cap M[G_\alpha] \} \) for \( G \ P^{\omega_1}-generic \) over \( M. \) This works since all reals in \( N = M[G] \) for \( G \ P^{\omega_1}-generic \) over \( M \) are caught at some countable stage.
Remark: It is easy to see that if \( V = L \) there is an \( X \subseteq \omega^\omega \) uncountable \( \Pi^1_1 \) set such that \( X \in L \) and \( X \times X \) is homeomorphic to \( X \). Also by absoluteness it is possible to make sure that for every \( A \in \Sigma^1_2 \) in \( \omega^\omega \), \( A \cap X \) is Borel in \( X \). This family of sets includes those obtained by the Souslin operation from Borel sets in \( X \).

Theorem 46. (MA) \( \exists X \subseteq 2^\omega \) \( \text{ord}(X) = \omega_1 \) and \( \forall A \in \Sigma^1_1 \) in \( 2^\omega \) \( \exists B \) Borel (\( 2^\omega \)) \( A \cap X = B \cap X \).

Proof.
Let \( B \) be the c.c.c. countably generated boolean algebra of Theorem 9 with \( K(B) = \omega_1 \).
\( B = \text{Borel}(2^\omega)/J \) for some \( J \) and \( \omega_1 \)-saturated \( \sigma \)-ideal in the Borel sets.

Lemma 47. If \( I \) is an \( \omega_1 \)-saturated \( \sigma \)-ideal in \( \text{Borel}(2^\omega) \) then \( B_I = \{ A \subseteq 2^\omega : \exists B \) Borel \( \exists C \in I \) \( (A \Delta B) \subseteq C \} \) is closed under the Souslin operation.

For a proof the reader is referred to [11], page 95.

By Theorem 14 MA implies there is \( X \subseteq 2^\omega \) a J-Luzin set. For any \( \alpha < \omega_1 \) there is \( A \in \Pi^0_\alpha \) so that for every \( B \in \Sigma^0_\alpha \), \( (A \Delta B) \notin J \), hence \( |(A \Delta B) \cap X| = |2^\omega| \), so \( A \cap X \notin B \cap X \), and thus \( \text{ord}(X) = \omega_1 \). If \( A \in \Sigma^1_1 \) then by Lemma 47 there is \( B \) Borel and \( C \) in \( J \) with
A ∆ B ⊆ C. Since |C ∩ X| < |2^ω| by MA ∃D ∈ Borel(2^ω) (A ∆ B) ∩ X = D ∩ X. So A ∩ X = (B ∆ D) ∩ X.

This suggests the following question:

Can you have X ⊆ 2^ω such that every subset of X is Borel in X and the Borel hierarchy on X has ω₁ distinct levels? The answer is no.

**Theorem 48.** If X ⊆ 2^ω and every subset of X is Borel in X then ord(X) < ω₁.

**Proof.**

Let X = \{x_α : α < κ\} and X_α = \{x_β : β < α\}

**Lemma 49.** If |X| ≤ κ, every subset of X is Borel in X, and R^K_ω₁ = P(κ × κ), then ord(X) < ω₁.

**Proof.**

Since every rectangle in X × X is Borel in X × X and R^K_ω₁ = P(κ × κ), every subset of X × X is Borel in X × X. Suppose for contradiction ∀α < ω₁ ∃H_α ⊆ X not \( \Pi^0_\alpha \) in X. Let H = \bigcup_{α < ω₁} x_α × H_α. For some α < ω₁, H is \( \Pi^0_\alpha \) in X × X. But then every cross section of H is \( \Pi^0_\alpha \) in X contradiction.

The proof of the theorem is by induction on |X| = κ.
For $\kappa = \omega_1$ it follows from Lemma 49 since $R_2^{\omega_1} = P(\omega_1 \times \omega_1)$.

For $\text{cof}(\kappa) = \omega$ it is trivial.

For $\text{cof}(\kappa) > \omega_1$:

$\forall \alpha < \kappa$ choose $\beta_\alpha$ minimal $< \omega_1$ so that every subset of $X_\alpha$ is $\prod_\beta^0$ in $X$ (we can do this since $X_\alpha$ is $\Pi_\beta^0$ in $X$ some $\beta < \omega_1$). Since $\text{cof}(\kappa) > \omega_1$ there exists $\alpha_0 < \omega_1$ such that for a final segment of ordinal less than $\kappa$, $\beta_\alpha = \alpha_0$. By Theorem 33 $R_1^\kappa = P(\kappa \times \kappa)$ so by Lemma 49 $\text{ord}(X) < \omega_1$.

For $\text{cof}(\kappa) = \omega_1$:

Let $\eta_\alpha + \kappa$ for $\alpha < \omega_1$ be an increasing continuous cofinal sequence.

**Lemma 50.** $\exists \beta_0 < \omega_1 \forall \alpha < \omega_1 X_{\eta_\alpha}$ is $\Pi_\beta^0$ in $X$.

**Proof.**

If $G \subseteq \kappa \times \kappa$ is the graph of a partial function then $G \in R_2^\kappa$ (Rao [21]). This is because if $f: D \to \kappa$ then viewing $X \subseteq \text{irrational real numbers}$ we have:

$(f(\alpha) = \beta)$ iff $(\alpha \in D \text{ and } \forall r \in Q(r < x_{f(\alpha)} \text{ iff } r < x_\beta))$

where $Q$ is the set of rational numbers.

Then $D = \{(\alpha, \beta): \alpha < \omega_1 \land \beta < \eta_\alpha\}$ is the complement in $\omega_1 \times \kappa$ of a countable union of graphs from $\kappa$ into $\omega_1$. 


Hence the set $\bigcup_{\alpha<\omega_1} \{x_\alpha\} \times X_{\eta_\alpha}$ is Borel in $X \times X$. Say it is $\Pi^0_{\beta_0}$. It follows that each $X_{\eta_\alpha}$ is $\Pi^0_{\beta_0}$.

For all $\lambda < \omega_1$ let $\beta(\lambda)$ be minimal so that every subset of $X_{\eta_\lambda}$ is $\Pi^0_{\beta(\lambda)}$ in $X$. If the hypothesis of Theorem 33 fails, then $\exists f : \omega_1 + \omega_1$ increasing so that for all $\lambda < \omega_1$ $\beta(f(\lambda)) < \beta(f(\lambda + 1))$. So for all $\lambda < \omega_1$ there is some $H_\lambda \subseteq X_{\eta_f(\lambda + 1)}$ which is not $\Pi^0_{\beta(f(\lambda))}$ in $X$. Since every subset of $X_{\eta_f(\beta)}$ is $\Pi^0_{\beta(\beta)}$ in $X$ we can assume $H_\lambda \subseteq (X_{\eta_f(\lambda + 1)} - X_{\eta_f(\lambda)})$. Let $H = \bigcup_{\lambda<\omega_1} H_\lambda$. Then $H$ is $\Pi^0_{\alpha_0}$ in $X$ for some $\alpha_0 < \omega_1$. But for each $\lambda$, $H_\lambda = H \cap (X_{\eta_f(\lambda + 1)} - X_{\eta_f(\lambda)})$, so each $H$ is $\Pi^0_{\alpha_0, \beta_0 + 1}$ in $X$, contradiction.

This ends the proof of Theorem 48.
Remark: Kunen has noted that Theorem 48 may be generalized to nonseparable metric spaces. Let $\mathcal{B}$ be a $\sigma$-discrete basis for $X$ and assume that every subset of $X$ is Borel in $X$. By using $\sigma$-discreteness it is easily seen that

$$\exists \mathcal{H} \subseteq \mathcal{B} \exists \beta < \omega_1 \text{ so that } \mathcal{B} - \mathcal{H} \text{ is countable and } \forall U \in \mathcal{H} \text{ ord}(U) \leq \beta.$$ 

But $Y = \{ x \in X: \forall U \in \mathcal{B} (x \in U \rightarrow U \notin \mathcal{H}) \}$ is separable and hence by the theorem $\text{ord}(Y) < \omega_1$, and so $\text{ord}(X) < \omega_1$.

As a partial converse of Theorem 33 we have:

**Theorem 51.** If $\kappa = |2^\omega|$, $\kappa^\kappa = \kappa$, and $R^\kappa_{\alpha_0} = P(\kappa \times \kappa)$, then there is $X \subseteq 2^\omega$ with $|X| = \kappa$ and every subset of $X$ of cardinality less than $\kappa$ is $\Pi^0_{\alpha_0}$ in $X$.

**Proof.**

Let $Z_\alpha$ for $\alpha < \kappa$ be all the subsets of $\kappa$ of cardinality less than $\kappa$. Put $Z = \bigcup_{\alpha < \kappa} \{ \alpha \} \times Z_\alpha$ and $W = \{ (\alpha, \beta): \alpha < \beta < \kappa \}$. Let $\{ A_n: n < \omega \}$ be closed under finite boolean combinations and $Z, W \in \{ A_n \times A_m: n, m < \omega \}_{\alpha_0}$. The map $F: \kappa \rightarrow 2^\omega$ defined by $(F(\alpha)(n) = 1$ iff $\alpha \in A_n)$ is 1-1 and the set $X = F'' \kappa$ has the required property. 

For any cardinal $\kappa$ let $R(\kappa)$ be the least $\beta < \omega_1$ such that $R^\kappa_\beta = P(\kappa \times \kappa)$ or $\omega_1$ if no such $\beta$ exists.
Theorem 52. It is relatively consistent with ZFC that $|2^\omega| = \omega_{\omega+1}$, for every $n \leq \omega$ $R(\omega_n) = 1 + n$, and $R(\omega_{\omega+1}) = \omega$. This can be generalized to show that for any $\lambda < \omega_1$, a limit ordinal it is consistent with ZFC that $R(|2^\omega|) = \lambda$.

Proof.

Let $M \models \text{"ZFC + MA + } 2^\omega = \omega_{\omega+1}\text{"}$ be countable and transitive. Let $\kappa = \omega_{\omega+1}$ and define $\mathbb{P}^\alpha$ for $\alpha \leq \kappa$ so that $\mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha * \mathbb{P}_{2+\beta+1}(X_\alpha, Y_\alpha)$ where $Y_\alpha \subseteq 2^\omega$, $Y_\alpha \in M$, $|Y_\alpha| = \omega_{\beta+1}$, and $\phi \models "X_\alpha \subseteq Y_\alpha"$. At limits take the direct limit. By dovetailing arrange that for any $G$ $\mathbb{P}^\kappa$-generic over $M$, $M[G] \models \text{"If } Y \subseteq 2^\omega, Y \in M\text{, and } |Y| = \omega_{\beta+1} \text{ for some } \beta < \omega\text{, then every subset of } Y \text{ is } \mathbb{P}^{\beta+1}$ in $Y$.

As in the proof of Theorem 34 given any $\tau$ a term for a subset of $\omega$, find in $M$, $H \subseteq 2^\omega$, $K \subseteq \kappa$ so that:

1. $|H| \leq \omega_{\beta_0}$, $|K| \leq \omega_{\beta_0}$.

2. $\forall n \in \omega \ Q \text{ decides } "n \in \tau"$.

3. $\forall \beta \in K \forall x \in H \ Q \text{ decides } "x \in X_\beta"$.

4. If $\alpha \in K$ and $|Y_\alpha| \leq \omega_{\beta_0}$ then $Y_\alpha \subseteq H$. 
Lemma 53. If \( H,K \) have property (3), (4) above then for any \( p \in \mathcal{P}^K \) and \( \beta \) with \( 1 \leq \beta < 2 + \beta_0 \), there is \( \hat{p} \) compatible with \( p \), \(|\hat{p}|(H) < \beta + 1\), \( \text{supp}(\hat{p}) \subseteq K \), and for any \( q \) if \(|q|(H) < \beta\), \( \text{supp}(q) \subseteq H \), and \( \hat{p} \) and \( q \) are compatible, then \( p \) and \( q \) are compatible.

Proof.
The proof of this is just like the proof of Lemma 35. To check that the \( \hat{p} \) gotten there is an element of \( \mathcal{P}^K \), note that if \((n,x) \in \hat{p}(\alpha)\) then \( x \in H \). Because if \( x \notin H \) and \( \alpha \in K \), then \(|Y_\alpha| > \omega_{\beta_0 + 1}\) because of (4). Say \(|Y_\alpha| = \omega_{\gamma + 1}\), so \( \mathcal{P}^{\alpha + 1} = \mathcal{P}^\alpha \ast \mathcal{P}_{2 + \gamma + 1}(X_\alpha, Y_\alpha) \) and \(|\langle n \rangle|_{2 + \gamma + 1} = 2 + \gamma > 2 + \beta_0 > \beta + 1\), but then it was thrown out, contradiction.

Lemma 54. Suppose \( H \) and \( K \) have properties (2), (3), and (4) for \( \tau \subseteq \omega \). Suppose \( 1 \leq \beta < 2 + \beta_0 \) and \( B(\upsilon) \) is a \( \mathcal{P}^0_\beta \) predicate with parameters from \( M \), \( p \in \mathcal{P}^K \) and \( p \models \text{"}B(\tau)\text{"} \). Then \( \exists q \in \mathcal{P}^K \) compatible with \( p \), \(|q|(H) < \beta\), \( \text{supp}(q) \subseteq K \) and \( q \models \text{"}B(\tau)\text{"} \).

Proof.
This follows from Lemma 53 just as in Theorem 34.

From Lemma 54 we have that:

(A) For any \( Y \subseteq 2^\omega \) with \( Y \subseteq M \) and \( n \) with \( 1 \leq n \leq \omega \) (\(|Y| = \omega_n \) iff \( Y \) is a \( Q_{2+n} \)-set). We claim that:
(B) For any $n < \omega$ there are $X, Y \subseteq 2^\omega$ with $|X| = |Y| = \omega_{n+2}$ so that if $U$ is the usual $\Pi^0_{n+2}$ set universal for $\Pi^0_{n+2}$ sets, then $U \cap (X \times Y)$ is not $\Sigma^0_{n+2}$ in the abstract rectangles on $X \times Y$.

To prove (B) just generalize the argument of Theorem 37, for $n = 0$ the argument is the same. Let $X \subseteq 2^\omega$ be in $M$ with $|X| = \omega_{n+2}$. Choose $K \subseteq \kappa$, $|K| = \omega_{n+2}$, and $K \in M$, so that for any $\alpha \in K$, $Y_\alpha = X$ and $\phi \models "X_\alpha = \phi"$. Let $Y = \{y_\alpha : \alpha \in K\}$ where $y_\alpha$ is the $\Pi^0_{n+2}$ code (with respect to $U$) for $G_{(0)}^\alpha$. To generalize the argument allow $I_X, J_X, I_\alpha, J_\alpha$ to have cardinality $\leq \omega_n$ and also whenever $Y \in J_X (\gamma \in J_\alpha)$ and $|Y_\gamma| \leq \omega_n$, then $Y_\gamma \subseteq I_X$ ($Y_\gamma \subseteq I_\alpha$).

In $M[G]$ for any $n < \omega$ $R(\omega_n) = 1 + n$. To see this, let $Y \subseteq 2^\omega$ with $Y \in M$ and $|Y| = \omega_{n+1}$. If $X \subseteq Y$ and $|X| \leq \omega_n$, then there is $Z \in M$ with $|Z| \leq \omega_n$ and $X \subseteq Z$. Because $M \models "MA"$, $Z$ is $\Pi^0_{n+2}$ in $Y$ and since $X$ is $\Pi^0_{n+2}$ in $Z$ by (A), we have $X$ is $\Pi^0_{n+2+n}$ in $Y$. By Theorem 33 $R_{n+2}^{\omega_n+1} = P(\omega_{n+1} \times \omega_{n+1})$. By (B) $n + 2$ is the least which will do.

Thus $R(\omega) = \omega$. To see that $R(\kappa) = \omega$ let $Y \subseteq 2^\omega$ with $Y \in M$ $|Y| = \kappa$, and every subset $Z \subseteq Y$ such that $|Z| < \kappa$ and $Z \in M$ is $\Pi^0_{\kappa}$ in $Y$ (see Theorem 17). In $M[G]$ every $Z \subseteq Y$ with $|Z| < \kappa$ is
Remark: It is easy to generalize Theorem 54 to show that for any \( \lambda < \omega_1 \) a limit ordinal and \( \kappa > \omega \) of cofinality \( \omega \), it is consistent that \( |2^\omega| = \kappa^+ \) and \( R(\kappa^+) = \lambda \).

**Theorem 55.** It is relatively consistent with ZFC that

(a) \( |2^\omega| = \omega_{\omega_1 + 1} \),
(b) for any \( \alpha < \omega_1 \) there is a \( Q_{\alpha} \) set.
(c) \( R(\omega_n) = n + 1 \) for \( n < \omega \),
(d) \( R(\omega_\lambda) = \lambda \) for \( \lambda < \omega_1 \) a limit ordinal,
(e) \( R(\omega_\lambda + n + 1) = \lambda + n \) for \( \lambda < \omega_1 \) a limit ordinal and \( n < \omega \).

The proof of this is an easy generalization of Theorem 54 and is left to the reader.

A set \( U \subseteq 2^\omega \times 2^\omega \) is universal for the Borel sets iff for every \( B \subseteq 2^\omega \) there exists \( x \in 2^\omega \) such that \( B = U_x = \{y: (y,x) \in U\} \).

**Theorem 56.** It is relatively consistent with ZFC that no set universal for the Borel sets is in the \( \sigma \)-algebra generated by the abstract rectangles in \( 2^\omega \times 2^\omega \).
Proof.
Let \( M \models \text{"ZFC + CH"} \) and let
\[
Q = \sum_{\beta < \omega_2} (E(P_\alpha(\phi, 2^\omega \cap M): \alpha < \omega_1)).
\]
Let \( G \) be \( Q \)-generic over \( M \), then in \( M[G] \) there is no set \( U \) universal for the Borel sets in the \( \sigma \)-algebra generated by the rectangles. Suppose \( G \) is given by
\[
(\gamma^\alpha_\beta: T_{\alpha+1}^{\times} \rightarrow 2^{\omega_1}: \alpha < \omega_1 \text{ and } \beta < \omega_2)
\]
where \( T_{\alpha+1} \) is the normal \( \alpha + 1 \) tree used in the definition of \( P_{\alpha+1} \) and \( G^{(0)} \) are the \( \Pi^0_\alpha \) sets determined by \( \gamma^\alpha_\beta \). Then as before we can easily get for each \( \alpha < \omega_1 \) that \( V^\alpha = \{ (x, \beta): x \in G^{(0)} \} \) is not \( \Sigma^0_{\omega_1} \) in the abstract rectangles on \( (2^\omega \times 2^\omega) \). Now suppose such a \( U \) existed and were \( \Sigma^0_{\omega_1} \) in the abstract rectangles on \( 2^\omega \times 2^\omega \).

Choose \( F: \omega_2 \rightarrow 2^\omega \) (necessarily 1-1) so that
\[
\forall \beta < \omega_2 \forall x \in 2^\omega \left( (x, \beta) \in V^\alpha \leftrightarrow (x, f(\beta)) \in U \right).
\]
If \( U \) is \( \Sigma^0_{\omega_1} \) in \( \{ A_n \times B_n: n < \omega \} \) then \( V^\alpha \) is \( \Sigma^0_{\omega_1} \) in \( \{ A_n \times f^{-1}(\beta_n): n < \omega \} \), contradiction. \( \square \)

Remarks:

(1) In [9] Kunen shows that if one adds \( \omega_2 \) Cohen reals to a model of GCH then no well ordering of \( \omega_2 \) is in \( R_{\omega_1}^{\omega_2} \).

(2) In [1] it is shown that if \( G \) is a countable field of sets with \( \text{Borel}(2^\omega) \subseteq G_{\omega_1} \), the order of \( G \) is \( \omega_1 \).
In the model of Theorem 56 for any countable $G$ and $\alpha < \omega_1$, Borel($2^\omega$) is not included in $G_\alpha$. This can be seen as follows. Let $G = \{A_n: n < \omega\}$ and let

$$\{s_n: n < \omega\} = T^s$$

where $T$ is a normal $\alpha$ tree. Define for any $y \in \omega$ and $s \in T$ the set $G_y^s$ as follows.

For $s = s_n$ let $G_y^s = A_y(n)$, otherwise $G_y^s = \bigcap \{\omega^n - G_y^n: n < \omega\}$. If $U = \{(x,y): x \in G_y^s\}$ then $U$ is "$\Pi_0^\alpha$" in the abstract rectangles and universal for all Borel sets, contradicting Theorem 56.

§5 Problems

Show:

(1) If $|X| = \omega_1$ then $X$ is not a $Q_\omega$ set.

(2) If $R_\omega^{\omega^2} = P(\omega_2 \times \omega_2)$ then there is $n < \omega$ with $R_n^{\omega^2} = P(\omega_2 \times \omega_2)$.

(3) If there exists a $Q_\omega$ set then there exists a $Q_n$ set for some $n < \omega$.

(4) If $R_1^{\omega^2} = P(\omega_2 \times \omega_2)$ and $|2^\omega| = \omega_2$ then $|2^{\omega_1}| = \omega_2$.

(5)* If there is a $Q_{\omega_1}$ set of size $\omega_1$ then every subset of $2^\omega$ of size $\omega_1$ is a $Q_2$ set.

(6) If $X$ is a $Q_\alpha$ set and $Y$ is a $Q_\beta$ set, then $2 \leq \alpha < \beta$ implies $|X| < |Y|$.
Show consistency of:

(7) \( \{ \alpha : X \subseteq 2^\omega \text{ ord}(X) = \alpha \} = \{1\} \cup \{ \alpha \leq \omega_1 : \alpha \text{ is even} \} \).

(8) \(|2^\omega| = \omega_3\) and for any \(X \subseteq 2^\omega\) if \(|X| = \omega_1\) then

\(X\) is a \(Q_7\) set, if \(|X| = \omega_2\) then \(X\) is a \(Q_{\omega+3}\) set, and if \(|X| = \omega_3\) then \(\text{ord}(X) = \omega_1\).

(9) For any \(\alpha \leq \omega_1\) there is a \(\Pi^1_1\) \(X\) with \(\text{ord}(X) = \omega_1\).

(10) For any \(X \subseteq 2^\omega\) if \(|X| \geq \omega_1\) then there is an \(X\)-projective set not Borel in \(X\).

(11) There is no \(G\) countable with \(\Sigma^1_1 \subseteq G\). (This is a problem of Ulam, see Fund. Math. 30 (1938), 365.)

*Answered by William Fleissner in the negative (to appear).*
REFERENCES


Vaught's conjecture [1] is that for any countable first order theory $T$, $\omega(T) \leq \aleph_0$ or $\omega(T) = 2^{\aleph_0}$ where, $\omega(T)$ is the number of nonisomorphic countable models of $T$.

Let $\sigma = (A, R_n)_{n<\omega}$ where each $R_n$ is $k_n$-ary relation. Define for $x, y \in A$:

$S(x, y)$ iff $x \neq y \cdot \exists n < \omega \exists x_1, \ldots, x_{k_n} \in A(R_n(x_1, \ldots, x_{k_n}) \wedge \wedge_{i, j}(x = x_i \wedge y = x_j))$. $(A, S)$ is the associated graph of $\sigma$ ($S$ is a symmetric, irreflexive binary relation). Define a metric $\delta(x, y)$ on $A$ as follows:

$\delta(a, b) = \begin{cases} \text{least } n \exists x_0, \ldots, x_n (x_0 = a \wedge x_n = b \wedge \wedge_{i < n} S(x_i, x_{i+1})) & \text{if no such } n \text{ exists.} \\ \end{cases}$

Define:

(1) $a$ is connected to $b$ iff $\delta(a, b) = n$ some $n < \omega$.

(2) $C \subseteq A$ is a component if it is a maximal connected subset.

(3) A loop is a set of points $\{x_0, \ldots, x_n\}$ with $n > 1$ such that $\wedge_{i < n} S(x_i, x_{i+1}) \wedge S(x_n, x_0) \wedge \bigwedge_{i \neq j} x_i \neq x_j$.

(4) $\omega(\sigma) =$ number of nonisomorphic elementary substructures of $\sigma$.

Theorem A. If $\sigma = (A, R_n)_{n<\omega}$ is a countable structure, $G = (A, S)$ the associated graph, and every component of $G$
contains only finitely many loops then \( \omega(\sigma) \leq \aleph_0 \) or \( \omega(\sigma) = 2^{\aleph_0} \).

Examples of \( \sigma \) satisfying the hypothesis are:

(1) \( \sigma = (A,R) \) where \( R \) is a binary relation which is a partial function on \( A \).

(2) \( \sigma = (A,R_n)_{n<\omega} \) where each \( R_n \) is a partial function on \( A \) and for each \( n \) and \( m \), \( R_n \) is equal to \( R_m \) on their common domain.

(3) If \( \sigma \) satisfies the hypothesis then so does any extension of \( \sigma \) by a countable number of unary predicates.

**Theorem B.** If \( T \) is a complete countable theory such that every countable model of \( T \) has the property that every component of its associated graph contains only finitely many loops then \( \omega(T) = 1, \aleph_0 \) or \( 2^{\aleph_0} \).

Theorem B was proved by myself and Leo Marcus [2] independently. Later M. Rubin pointed out that the fact that \( (\omega(T) > 1 \Rightarrow \omega(T) \geq \aleph_0) \) can be obtained as a corollary of a theorem of Lachlan [3] since every such theory is super-stable. The author of this fact is unknown to me. Note that if \( M < N \models T \) then for any \( a,b \in N - M \) if \( \hat{c} \in M \) is the "closest" element of \( M \) to \( a,b \) then
\[ <N,a,c> \equiv <N,b,c> \quad \text{and} \quad <N,a,c>_{c \in M} \equiv <N,b,c>_{c \in M}. \]

(This is easily shown by using Lemma 1 and Ehrenfeucht games.) But this shows there are at most \(2^{\aleph_0} \cdot |M|\) 1-types over \(M\). Similar argument works for \(n\)-types. Note that if a countable theory \(T\) fails to have an \(\omega\)-saturated countable model then \(\omega(T) = 2^{\aleph_0}\), hence the rest of Theorem B follows from Theorem A by determining \(\omega(\sigma)\) for \(\sigma\) countable \(\omega\)-saturated. It is also not hard to show that the number of non-isomorphic elementary extensions of a model satisfying the hypothesis of Theorem A is 1, \(\aleph_0\), 2 \(\aleph_0\).

**Theorem C.** There is a \(\theta\) a \(PC(L_{\omega_1 \omega})\) sentence in one unary operation such that \(\omega(\theta) = \aleph_1\).

This disproves the main result of Stanley Burris [4] by showing that the quantifier ranks of Scott sentences of a countable unary operation are arbitrarily high. John Steel [5] has proved Vaught's conjecture for \(L_{\omega_1 \omega}\) sentences in one unary operation.

Matatyahu Rubin proved Vaught's conjecture for theories of a linear order [8] and more recently for \(L_{\omega_1 \omega}\) sentences of a linear order [9]. In my abstract [11] I mistakenly stated Theorem C for \(PC(L_{\omega \omega})\).

**Question:** Does there exist a \(PC(L_{\omega \omega})\) sentence \(\theta\) in
one unary operation with $\omega(\theta) = \mathcal{K}_1$?

For any $(L,\prec)$ a linear order define the following unary operation $(U_L,F_L)$

$$U_L = \{(a_0,\ldots,a_{n-1}): n < \omega, a_0 > a_1 > \ldots > a_{n-1} \forall i < n \text{ } a_i \in L\}$$

$$F_L(\prec) = \prec$$

$$F_L(\langle a_0,\ldots,a_n\rangle) = \langle a_0,\ldots,a_{n-1}\rangle$$

Claim: If $L = L_1 + L_2$ and $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ are countable linear orders, $L_1$ and $\mathcal{L}_1$ are isomorphic well orders, and either $L_2$ and $\mathcal{L}_2$ are both empty or they are both non-empty and have no least element then $(U_L,F_L)$ is isomorphic to $(U_\mathcal{L},F_\mathcal{L})$.

Thus $\theta = \{(U,F): \exists L \text{ countable linear order } \langle U,F \rangle = U_LF_L \}$ is $PC(L_\omega\omega)$ and $\omega(\theta) = \mathcal{K}_1$. □

We only prove Theorem A for $\mathcal{O}_1 = (A,R,a)$ where $R$ is binary, symmetric, irreflexive; and $a$ is finitely many constants, since it is easy to generalize.

Definition: (1) for $\mathcal{O}_1$ having a distinguished constant $\overline{a}$ let $\mathcal{O}_1^\overline{a} = \{a \in A: \delta(a,\overline{a}) < n\}$.
(2) $\mathcal{G}_1 \equiv_n \mathcal{G}$ iff Player II has a winning strategy in the Ehrenfeucht game of length $n$ [6].

Our main lemma is the following, its proof is on p.88.

Lemma 1. If $\mathcal{G}$ and $\mathcal{G}'$ are connected with distinguished constants then $(\forall n < \omega \mathcal{G}_n \equiv_n \mathcal{G}'_n) \implies \mathcal{G} \equiv \mathcal{G}'$.

Lemma 2. If $\forall \mathcal{C}$ component of $\mathcal{G}$ countable structure $(\omega(\mathcal{C}) \leq \aleph_0$ or $\omega(\mathcal{C}) = 2^{\aleph_0}$), then $\omega(\mathcal{G}) \leq \aleph_0$ or $\omega(\mathcal{G}) = 2^{\aleph_0}$.

Proof:

Note that from Lemma 1 if $\mathcal{G} \leq \mathcal{G}$, then the components of $\mathcal{G}$ are elementary substructures of the corresponding components of $\mathcal{G}$. If $\omega(\mathcal{C}) = 2^{\aleph_0}$ some $\mathcal{C}$ a component of $\mathcal{G}$ then using Ehrenfeucht games we see $\omega(\mathcal{G}) = 2^{\aleph_0}$.

Otherwise let $\{\mathcal{C}_n : n < \omega\}$ be pairwise nonisomorphic so that $\forall \mathcal{G} \leq \mathcal{C}$ a component of $\exists n \mathcal{C}_n = \mathcal{G}$. For $k : \omega + \omega + 1$ let $\mathcal{G}_k$ be a structure (obtained continuously from $k$) with exactly $k(n)$ copies of $\mathcal{C}_n$ for each $n$ and universe subset of $\omega$.

$X = \{k \in (\omega+1)^\omega : \mathcal{G}_k \text{ can be elementarily embedded into } \mathcal{G}\}$

$X$ is a $\Sigma^1_1$ set and $|X| = \omega(\mathcal{G})$ so by a classical theorem of descriptive set theory [7] $\omega(\mathcal{G}) \leq \aleph_0$ or $\omega(\mathcal{G}) = 2^{\aleph_0}$.
Note: $(\forall a \in A \omega(<\sigma_1, a>) \leq \aleph_0) \rightarrow \omega(\sigma_1) \leq \aleph_0$.

$(\exists a \in A \omega(<\sigma_1, a>) = 2 \aleph_0) \rightarrow \omega(\sigma_1) = 2 \aleph_0$.

If $\sigma_1$ is connected and $Y \subseteq A$ is finite and contains all of $\sigma_1$'s loops then define $\sigma_1(y)$ for $y \in Y$

$\sigma_1(y) = \{a \in A: a$ is connected to $y$ by a path which only intersects $Y$ at $y\}$. By Lemma 1 note that for $\nu \leq \sigma_1$

$(<\nu, y>_{y \in Y} \leq <\sigma_1, y>_{y \in Y}$ iff $\nu(y) \leq \sigma_1(y)$ for $y \in Y$.

Hence it is enough to count the number of elementary substructures of a tree. Define $\sigma_1$ is a tree iff countable, connected, has no loops, and has a distinguished constant $0$. From now on all structures are trees until p. 91.

Examples:

I) Let $T_n = \ldots \ldots$ \hspace{1cm} $n \leq \omega$

$T_{n,m} = \ldots \ldots$ \hspace{1cm} $n,m \leq \omega$

Consider $T = \ldots \ldots$

where for each $n,m \leq \omega$ infinitely many of the $S_i$ are $T_{n,m}$. $\omega(T) = 2 \aleph_0$ is shown by Lemma 3.

II) Let $S = \ldots \ldots$ \hspace{1cm} $\omega(S) = \aleph_0$

a) \hspace{1cm} \ldots \ldots
Each of these has $2\lambda_0$ nonisomorphic elementary substructures.

III) To illustrate Lemma 6:

Extend $<$ on $\omega$ to $\omega \cup \{\infty\}$ by $n < \infty \forall n < \omega$ and $\infty < \infty$. Let $U = \{(a_0, \ldots, a_{n-1}): n < \omega, a_0 > a_1 > \ldots > a_{n-1}, a_i \in \omega \cup \{\infty\}\}$. If $F$ is the projection function on $U$ ($F((a_0, \ldots, a_{n+1})) = (a_0, \ldots, a_n)$) then $\omega(F) = \lambda_0$.

Definitions:

1. $a$ is below $b$ iff $b$ lies on the unique shortest path connecting $a$ to $0$.

2. $\sigma_l(a)$ is the tree with universe $\{b \in A: b$ is below $a\}$ and distinguished constant $a$.

3. $P(\sigma_l) = \{a \in A: \delta(a,0) = 1\}$ and for $a \in A$
   $P(a) = P(\sigma_l(a))$.

4. for $X \subseteq P(\sigma_l)$ $\sigma_l[X]$ is the tree with universe $0$ and elements of $A$ below things in $X$ and with
distinguished constant $0$.

(5) for $x_n \in P(\sigma)$ and $y \in P(\sigma)$, $x_n + y$ iff $x_n \neq x_m$ for $n \neq m$ and $t_p(x_n, \sigma) + \text{typ}(y, \sigma)$, i.e. $\forall \forall(v)$ first order $\exists N \forall n \geq N \ (\sigma \models \psi(x_n) \iff \sigma \models \psi(y))$.

**Lemma 3.** If $X \uplus Y = P(\sigma)$ are disjoint and $\forall y \in Y \ (\sigma'[x_n \in X] \models y)$, then $\sigma'[y] \models y$.

**Proof:**

It is easy to find $X_y = \{x_n^y : n < \omega\} \subseteq X$ for $y \in Y$ disjoint so that $x^y \models y$ for each $y \in Y$.

**Claim:** $\forall n_0 < \omega \forall y \in Y \sigma_{n_0}[X_y] \leq \sigma_{n_0}[X_y \cup \{y\}]$

**Proof:**

Let $\mathcal{U} = \sigma_{n_0}$ and $X_y = \{x_n : n < \omega\}$. Clearly $\ast$ holds for $\mathcal{U}$ in place of $\sigma$, hence we know from the basic lemma on Ehrenfeucht games [6] that $\forall n < \omega \exists N < \omega \forall m > N \mathcal{U}(x_m) \models \mathcal{U}(y)$. Given $\mathcal{A} \subseteq \mathcal{U}[X_y]$ and $n_1 < \omega$, choose $N$ sufficiently large so that $\mathcal{A} \subseteq \mathcal{U}[\{x_n : n < N\}]$ and for $m > N \mathcal{U}(x_m) \models n_1 \mathcal{U}(y)$. Now patch together appropriate strategies for Player II as follows:

Claim
From Lemma 1 and Claim, $\sigma [X_y] \leq \sigma [X_y \cup \{y\}]$ for each $y \in Y$, hence by an easy Ehrenfeucht game argument $\sigma [X] \leq \sigma \cdot \blacksquare$

**Definition:** $\sigma$ is simple iff $\forall a \in A$ only finitely many nonprincipal types in $Th(\sigma(a))$ are realized in $P(a)$.

**Note:** By using Lemma 3 if $\sigma$ is not simple then $\omega(\sigma) = 2^{|X_0|}$.

**Definition:** Given $(\mathcal{P}_a : a \in A)$ such that $\mathcal{P}_a \leq \sigma(a)$ for each $a$ the fusion of $(\mathcal{P}_a : a \in A)$ is the tree $\mathcal{P}$ with $0^\mathcal{P} = 0^\sigma$ and universe $\{b : \text{for all } a \text{ between } 0 \text{ and } b, b \in \mathcal{P}_a\}$.

**Lemma 4.** Given $(\mathcal{P}_a : a \in A)$ with $\mathcal{P}_a \leq \sigma(a)$ all $a$ and $\mathcal{P}$ the fusion then $\mathcal{P} \leq \sigma \cdot \blacksquare$

**Proof:**

By Lemma 1 we may assume $\sigma = \sigma_n$ for some $n < \omega$.

Now prove it by induction on $n$. Thus $\mathcal{P}(b) \leq \sigma(b)$ for all $b \in P(\sigma)$, hence $\mathcal{P}(b) \leq \mathcal{P}_0(b) \forall b \in P(\mathcal{P}_0)$ and by an easy Ehrenfeucht game argument $\mathcal{P} \leq \mathcal{P}_0 \leq \sigma \cdot \blacksquare$

**Definition:** If $\sigma$ is simple let $\mathcal{P}_a^{pr} = \sigma(a)[\{x : tp(x, \sigma(a)) \text{ is principal}]$ for each $a \in A$, and $\sigma^{pr}$ be the fusion of $\mathcal{P}_a^{pr} : a \in A$. By Lemma 3 $\mathcal{P}_a^{pr} \leq \sigma(a)$
and by Lemma 4 $\alpha^\text{Pr} \leq \alpha_1$.

Lemma 5. If $\alpha^\text{Pr} = \alpha_1$ then $\omega(\alpha_1) = 1$.

The proof is straightforward and left to the reader.

Definitions:

$N(a) = \{x \in P(\sigma_1(a)) : \text{tp}(x, \sigma_1(a)) \text{ is nonprincipal}\}$

$L = \{a \in A : N(a) \neq \emptyset\}$

$T = \{b \in A : \exists a \in L \ b \text{ lies on the unique shortest path connecting } a \text{ to } 0\}$.  

Lemma 6. If $L = \{a_n : n < \omega\}$ and $\forall n N(a_n) = \{b_n\}$ and $a_{n+1} \in \sigma_1(b_n)$ then $\omega(\sigma_1) \leq \kappa_0$.

Proof:

Let for each $n < \omega$ $\mathcal{G}_n = \sigma_1 - \sigma_1(b_n)$ then these are all the nonisomorphic elementary substructures of $\sigma_1$.

Definition: (1) $[T]$ is the set of infinite branches of $T$.

(2) $a \in A$ isolates $f \in [T]$ iff $\sigma_1(a)$ is as in the hypothesis of Lemma 6 with $a \in f$.

Lemma 7. If $\sigma_1$ is simple and $\exists f \in [T]$ such that no $a \in A$ isolates $f$ then $\omega(\sigma_1) = 2^{\kappa_0}$. 
Proof:
Choose $a_n \in L$ and $b_n \in N(a_n)$ for $n < \omega$ as follows:
Having chosen them for $m < n$, let $c$ be any element of $f$ lower than any of the $a_m$ and $b_m$ for $m < n$. Since $c$ does not isolate $f$, let $a_n \in O_L(c) \cap L$ and $b_n \in N(a_n)(b_n \notin f)$.

Let $B = \{c : c$ is strictly between some $b_n$ and $0\}$
For $a \notin B$ let $\nu_a = (\sigma)(a)^P$
for $a \in B$ let $\nu_a = \sigma(a)[X_a]$ where
$X_{a_n} = \{x : tp(x, \sigma(a_n))$ is principal\}
$\cup \{b_n\}$
$X_d = \{x : tp(x, \sigma(c))$ is principal\}
$\cup \{P(\sigma(c)) \cap B\}$ if $d \notin a_n$ any $n < \omega$.

If $L$ is the fusion of the $\nu_a$'s then $L \leq \sigma 1$. For any $n < \omega$ note that at most two $x \in C$ such that
$\delta(x, 0) = \delta(a_n, 0)$ and $N(x)L \neq \emptyset$. For any $X \leq \omega$ let
$L_X \leq L$ be gotten by fusion so that $\forall n < \omega(b_n \in |L_X|$
iff $n \in X)$.

$x \neq x' \rightarrow L_X \neq L_{X'}$.  \[ \blacksquare \]

Lemma 8. If $\forall a \in P(\sigma) \omega(\sigma(a)) \leq \lambda_0^\delta$ or $\omega(\sigma(a)) = 2^{\lambda_0^\delta}$ then $\omega(\sigma) \leq \lambda_0^\delta$ or $\omega(\sigma) = 2^{\lambda_0^\delta}$.
The proof of this is similar to the proof of Lemma 2.

Lemma 9. If $\sigma$ is a tree then $\omega(\sigma) \leq \kappa_0$ or $\omega(\sigma) = 2^{\kappa_0}$.

**Proof:**
If $\sigma$ is not simple then $\omega(\sigma) = 2^{\kappa_0}$ by using Lemma 3.

Define $D(T) = \{x \in T : x$ does not isolate any $f \in [T]\}$.

By Lemma 7 if $D(T)$ is not well-founded ($[D(T)] \neq \emptyset$) then $\omega(\sigma) = 2^{\kappa_0}$. If $D(T) = \emptyset$ then by Lemma 5 or 6 $\omega(\sigma) \leq \kappa_0$. Hence we may assume $D(T)$ is well-founded and then the Lemma is proved by induction on the rank of $D(T)$ by using Lemma 8.

It remains only to prove Lemma 1. We no longer consider just trees.

Lemma 10. If $\sigma$ is connected with distinguished constant then $\forall n < \omega \forall \phi(x,y) \exists N \geq n \exists f \in \Gamma$ finite $\forall a \in \sigma - \sigma_N \exists \phi^*(y) \in \Gamma \forall b \in \sigma_n(\sigma_n \models \phi(a,b))$ iff $\sigma_n \models \phi^*(b)$.

**Proof:**
The proof is by induction on the logical complexity of $\phi(x,y)$. For the atomic case put $N = n + 2$ and $\Gamma = \{T,F,x_1 = x_2,R(x_1,x_2)\}$. On the induction step $\tau$, these are both easy.
$\exists z \phi(x, z, y)$

By induction $\exists \Gamma_1 \exists N_1 \geq n$ such that $\forall a \in \sigma_1 - \sigma_{N_1}$ $\exists \sigma(y) \in \Gamma_1 \forall \delta \in \sigma_n \ (\sigma_1 \models \phi(\bar{a}, a, \delta))$ iff $\sigma_{N_1} \models \sigma(\delta)$.

Also by induction $\exists \Gamma_2 \exists N_2 \geq N_1$ such that $\forall a \in \sigma_1 - \sigma_{N_2}$ $\exists \tau(z, y) \in \Gamma_2 \forall b \delta \in \sigma_{N_1}$ $(\sigma_1 \models \phi(a, b, \delta))$ iff $\sigma_{N_2} \models \tau(b, \delta))$. Let $N = N_2$ and

$\Gamma = \{ \omega \in F \cap N_1(y) \exists z \in \sigma_{N_1} \tau(z, y) \mid F \in \Gamma_1, \tau \in \Gamma_2 \}$. These work since given $a \in \sigma_1 - \sigma_{N_2}$ let

$F = \{ \sigma(y) \in \Gamma_1 : \exists a \in \sigma_1 - \sigma_{N_1} \forall \delta \in \sigma_n (\sigma_1 \models \phi(\bar{a}, a, \delta))$ $\leftrightarrow \sigma_{N_1} \models \sigma(\delta)) \}$ and $\tau(z, y)$ so

$\forall b \delta \in \sigma_{N_1} (\sigma_1 \models \phi(a, b, \delta) \leftrightarrow \sigma_{N_2} \models \tau(b, \delta))$. Let

$\phi*(y) = \omega \in F \cap N_1(y) \exists z \in \sigma_{N_1} \tau(z, y)$.

Remark: Lemma 10 was motivated by the main lemma in Feferman-Vaught [10].

Lemma 11. If $\sigma_1$ is connected with distinguished constant then $\forall \phi(x, y) \forall n < \omega \exists \sigma < \omega \forall \delta \in \sigma_n$ if $\sigma_1 \models \exists x \phi(x, \delta)$ then $\exists a \in \sigma_1 - \sigma_{N_1}$ $\forall b \in \sigma_1 \models \phi(a, \delta)$.

Proof:

Let $N_1, \Gamma$ be from Lemma 10 for $\phi(x, y)$ and $n$. Define:

$\phi^*(y) \in \Gamma$ is a testing formula for $a \in \sigma_1 - \sigma_{N_1}$ if $\forall \delta \in \sigma_n (\sigma_1 \models \phi(a, \delta) \leftrightarrow \sigma_{N_1} \models \phi^*(\delta))$. Choose $N \geq N_1$, $N < \omega$ so that $\forall a \in \sigma_1 - \sigma_{N_1}$ if $\phi^*(y) \in \Gamma$ is a testing formula for $a$ then there exists $\exists a \in \sigma_1$ so that
\( \phi^*(\overline{y}) \) is a testing formula for \( a' \). This \( N \) works because
\[
\models \phi(a,b) \iff \models \phi^*(\overline{b}) \iff \models \phi(a',\overline{b}) \text{ some } a' \in \mathcal{A}_N \text{ with same testing formula } \phi^*(\overline{y}) \text{ as } a.
\]

**Lemma 12.** If \( \models \) is connected with a distinguished constant and \( \models \equiv \mathcal{L} \) then \( \bigcup_{n \in \omega} \mathcal{L}_n \models \mathcal{L} \).

**Proof:**
If \( \overline{b} \in \mathcal{L}_n \) and \( \phi(x,\overline{y}) \) are given then taking \( N < \omega \) from Lemma 11, \( \models \quad \forall \overline{y} \in \mathcal{L}_n (\exists x \phi(x,\overline{y}) \iff \exists x \in \mathcal{A}_N \phi(x,\overline{y}))'. \) So if \( \mathcal{L} \models \exists x \phi(x,\overline{b}) \) then \( \exists b : \mathcal{L}_N \models \phi(b,\overline{b}). \)
By Tarski's criterion we are done.

Let \( (HC,\varepsilon) \prec M \) such that \( \omega^M \) is nonstandard.
We assume \( \mathcal{A}, \mathcal{L} \in HC. \) Let \( \mathcal{A}^* \) be the structure determined by \( M \) corresponding to \( \mathcal{A} \) and \( \mathcal{A}^*_n = \bigcup_{n \omega} \mathcal{A}^*_n. \) Let \( n^* \in \omega^M - \omega \) and \( M \models \quad 's \text{ is a strategy for player II in the Ehrenfeuch} \text{t game of length } n^* \text{ played between } \mathcal{A}^*_n \text{ and } \mathcal{L}^*_n'. \)
Since \( n^* \) is nonstandard the strategy \( s \) gives a back and forth property to show \( \mathcal{A}^*_n \equiv \mathcal{L}^*_n \) (if player I plays \( a \in \mathcal{A}^*_n \) then \( s \) must respond with \( b \in \mathcal{L}^*_n \)).
By Lemma 12 \( \mathcal{A}^*_n \subseteq \mathcal{A}^* \) and \( \mathcal{L}^*_n \subseteq \mathcal{L}^* \) and also \( \mathcal{A} \leq \mathcal{A}^* \) and \( \mathcal{L} \leq \mathcal{L}^* \) so \( \mathcal{A} \equiv \mathcal{L} \).
REFERENCES


PART III. THERE ARE NO Q-POINTS IN LAVER'S MODEL
FOR THE BOREL CONJECTURE

All ultrafilters are assumed nonprincipal and on \( \omega \).

Define:

1. \( U \) q-point (also called rare [C]) iff \( \forall (P_n : n < \omega) \)
a partition of \( \omega \) into finite sets \( \exists A \in U \forall n |A \cdot P_n| < 1 \).

2. \( U \) p-point iff \( \forall (P_n : n < \omega) \) partition of \( \omega \) either
   \( \exists n P_n \in U \) or \( \exists A \in U \forall n |A \cdot P_n| \) finite.

3. \( U \) is Selective (also called Ramsey) iff \( \forall (P_n : n < \omega) \)
   partition of \( \omega \) either \( \exists n P_n \in U \) or
   \( \exists A \in U \forall n |A \cdot P_n| < 1 \).

4. \( U \) is semiselective iff Given \( A_n \in U \exists f \in \omega^\omega \)
   \( \forall n f(n) \in A_n \) and \( f'' \omega \in U \).

5. \( U \) is semi q-point (also called rapid [C]) iff
   \( \forall f \in \omega^\omega \exists g \in \omega^\omega \forall n f(n) < g(n) \) and \( g'' \omega \in U \).

It is easily seen:

Selective = p-pt + q-pt
semiSelective = p-pt + semi q-pt

Define: \( f < g \) iff \( \exists n \forall m > n (f(m) < g(m)) \)
\( \mathcal{F} \subseteq \omega^\omega \) is dominant family iff \( \forall f \in \omega^\omega \exists g \in \mathcal{F} \ f < g \).

**Theorem (1)** (Ketonen [Ke]) If \( \exists \mathcal{F} \) dominant \( |\mathcal{F}| = 2^{\aleph_0} \)
then \( \exists \) a p-pt.

**Theorem (2)** (Mathias, Taylor) If \( \exists \mathcal{F} \) dominant \( |\mathcal{F}| = \kappa \),
then \( \exists \) a q-pt.
Kunen [Kul] showed that adding $\aleph_2$ random reals to a model of $\text{ZFC} + \text{GCH}$ gives a model with no semiselective ultrafilters. More recently he showed [Ku2] that if you first add $\aleph_1$ Cohen reals (then the random reals) then the resulting model has a $p$-pt. In either case one has a dominant family of size $\aleph_1$ so there is a $q$-pt.

The following are equivalent:

1) $U$ is semi $q$-pt.

2) $\forall (P_n : n < \omega) (\forall n P_n \text{ finite}) \rightarrow \exists A \in U \forall n |A \cap P_n| < n.$

3) $\exists h \in \omega \forall (P_n : n < \omega) (\forall n P_n \text{ finite}) \rightarrow \exists A \in U \forall n |A \cap P_n| < h(n).$

Proof.

1) $\Rightarrow 2)$ Let $f(n) = \sup (\bigcup_{m \leq n} P_m) + 1.$ Suppose $g(n) > f(n)$ all $n$ then $P_n \cap g'' \omega \subseteq \{g(0), \ldots, g(n-1)\}.$

3) $\Rightarrow 1)$ Assume $f$ increasing. Choose $n_0 < n_1 < n_2 < \ldots$ so that $h(k+1) < n_k.$ Let $P_k = f(n_k)$ and let $Y \in U$ be so that $|Y \cap P_k| < h(k).$ Then for each $m > n_0$

$|Y \cap f(m)| < m,$ since if $n_k \leq m < n_{k+1}$ then $|Y \cap f(n_{k+1})| < h(k+1) < n_k < m.$ Hence if $g \in \omega$

enumerates $Y - f(n_0 + 1)$ in increasing order then $\forall n f(n) < g(n).$

Define $U \times V = \{A \subseteq \omega \times \omega : \{n : (m : (n,m) \in A) \in V\} \in U\}.$

Whilst $U \times V$ is never a $p$-pt. or a $q$-pt. nevertheless:
$U \times V$ is semi q-pt. iff $V$ is semi q-pt.

Proof.

![Upper diagonal in $U \times V$]

$(\Rightarrow)$

Given $P_k \subseteq \omega$ finite let $P_k^* = \{\langle n, m \rangle : m \in P_k \land n \leq m\}$. Choose $Z \subseteq U \times V$ so that $\forall k |Z \cap P_k^*| \leq k$. Let $n \in \omega$ so that $Y = \{m \geq n : (n, m) \in Z\} \subseteq V$ then $\forall k |Y \cap P_k| \leq k$.

(More generally if $U = V$ and $U$ semi-q-pt. and $f$ finite to one then $V$ semi q-pt.)

$(\Leftarrow)$

Given $P_k \subseteq \omega \times \omega$ finite choose $n_k$ increasing so that $P_k \subseteq n_k^2$. Let $Y \subseteq V$ so that $\forall k |n_k \cap Y| \leq k$. Let $Z = \bigcup_{k<\omega} \{k\} \times \{m : m \in Y \land m \geq n_k\}$ then $Z \cap P_k \subseteq Z \cap n_k^2 \subseteq k \times (n_k \cap Y)$ which has cardinality $\leq (k + 1)^2$.

Theorem. In Laver's model $N$ for the Borel conjecture $[L]$ there are no semi q-pt's.

Proof.

Some definitions from $[L]$:

(1) $T \in \mathcal{F}$ iff $T$ subtree of $\omega^{<\omega}$ with the property that
\exists s_T \in T \text{ (called the stem of } T) \text{ so that } \forall t \in T
\begin{align*}
t \leq s_T \text{ or } s_T \leq t \text{ and for all } t \geq s_T & \Rightarrow \text{ there are infinitely many } \hat{t} \in T \text{ immediately below } t \text{ (} t \text{ is immediately below } s = (k_0, \ldots, k_n) \text{ iff } \exists k_{n+1} \\
t = (k_0, \ldots, k_n, k_{n+1}) & \Rightarrow T \supseteq \hat{T} \text{ iff } \hat{T} \text{ is a subtree of } T.
\end{align*}

(2) \( T_s = \{t \in T : s \leq t \cup t \leq s\} \).

(3) \( T^0 \supseteq \hat{T} \) iff \( T \supseteq \hat{T} \) and they have the same stem.

Lemma 1. Given \( T \in \mathcal{J} \) and for each \( s \in T - \{\emptyset\} \)
\[ F_s \subseteq [k_n, k_{n+1}) = \{x : k_n \leq x < k_{n+1}\} \]
where
\[ s = (k_0, \ldots, k_n, k_{n+1}) \text{ (} F_{\langle n \rangle} \subseteq [0, n] \text{) and } \forall s \in T, \exists N < \omega \ \forall t \text{ immediately below } s \text{ in } T \mid F_t \mid \leq N. \]
Then letting
\[ H_{T_0} = \bigcup_{s \in T} F_s \text{ for any } \hat{T} \supseteq T, \text{ we can find } T_0, T_1^0 \supseteq T \text{ so that } \]
\[ H_{T_0} \cap H_{T_1} \text{ is finite.} \]

Proof.

We may as well assume stem of \( T \) is \( \emptyset \).

Given \( Q \) any infinite family of sets of cardinality
\[ \leq N < \omega \text{ there exists } G, |G| \leq N, \exists Q \subseteq Q \text{ infinite so that } \]
\[ \forall F, F \in Q \text{ (} F \wedge F \subseteq G. \text{ Now trim } T \text{ to obtain } \hat{T} \supseteq T \]
and \( \forall s \in \hat{T}, \exists G_s \subseteq [k_n, \omega] \text{ finite } (s = (k_0, \ldots, k_n)) \) and
for all \( t, \hat{t} \text{ immediately below } s \text{ in } \hat{T}, (F_t \wedge F_{\hat{t}}) \subseteq G_s. \)

Build two sequences of finite subtrees of \( \hat{T} \): 
\[ T_0^n \subseteq T_n^0, \ldots \]
\[ T_1^n \subseteq T_n^1, \ldots \]
so that \([\bigcup_{s \in T_0} (F_s \cup G_s)] \cap [\bigcup_{s \in T_1} (F_s \cup G_s)] \subseteq G_\phi\)

and \(\bigcup_{n<\omega} T_i^n = T_i \supseteq \hat{T}\) for \(i = 0,1\)

This is done as follows: Suppose we have \(T_0^n, T_1^n\) and we're presented with \(s \in T_0^n\) and asked to add an immediate extension of \(s\) to \(T_0^n\). Then since

\[(F_t - G_s : t \text{ immediately below } s \text{ in } \hat{T})\]

is a family of disjoint sets and \(G_t \subseteq [k_n, \omega]\) where \(t = (k_0, \ldots, k_n)\) we can find infinitely many \(t\) immediately below \(s\) in \(\hat{T}\) so that

\[\left[(F_t - G_s) \cup G_t\right] \cap [\bigcup_{s \in T_0} (F_s \cup G_s)] = \emptyset\]

The above is a double fusion argument.

Some more definitions from [L]:

(1) Fix a natural \(\omega\)-ordering of \(\omega^{<\omega}\) and for any \(T \in \mathcal{F}\), transfer it to \(\{t : t \supseteq s_T \wedge t \in T\}\) in a canonical fashion. \(\hat{T}^n \supseteq T\) means the first \(n\) elements in this order on \(\{t : t \supseteq s_T \wedge t \in T\}\) are still in \(\hat{T}\).

(2) The p.o. \(\mathbb{P}_{\omega_2}\) is the \(\omega_2\) iteration of \(\mathcal{F}\) with
countable support \((p \models \alpha \in \mathcal{M}[\mathcal{G}_\alpha])\), all \(\alpha\) and support \((p) = \{\alpha : p(\alpha) \neq \omega\} \) is countable).

(3) For \(K\) finite and \(n < \omega\)
\[ p \models n \geq q \iff [p \models q \forall \alpha \in K p \models \alpha \models n \geq q(\alpha)]. \]

**Lemma 2.** Let \(f\) be a term denoting the first Laver real and \(t\) any term. If \(p \models \omega \models \forall n f(n) < \tau(n) \land \tau\) increasing" then \(\exists Z_0, Z_1, Z_0 \cap Z_1\) finite, \(\exists p_0, p_1 \models p\) and \(p_i \models \tau^n \models \omega \leq Z_i\) for \(i = 0, 1\).

**Proof.**

Construct a sequence \(p \models \omega \models p \models \omega \models p_{n+1} \equiv \omega \), so that
\[ \bigcup_{n<\omega} K_n = \bigcup_{n<\omega} \text{support}(p_n). \quad 0 \in K_0. \]

Having gotten \(p_n\) let \(s = (k_0, \ldots, k_m)\) be the \(n\)th member of \(\{t \in P_n(0) : t \leq \text{the stem of } P_n(0)\}\) \((s = p_n(0) < n\) in Laver's notation).

Fix \(t = (k_0, \ldots, k_m, k_{m+1})\) in \(P_n(0)\). Then for each \(i \leq m + 1\)
\[ p_t = \varphi p_n(0)t \models p_n(1, \omega_2) \models \tau(i) \geq k_{m+1} \land \omega \land \tau(i) = \omega. \]

Hence by applying Lemma 6 of \([L]\) \(m + 2\) many times we can find \(q_t \models \omega \models p_t\) and \(F_t \subseteq [k_m, k_{m+1}]\) such that
\[ |F_t| \leq (m + 2)(n + 1)|K_n| \]
and
\[ q_t \models \tau^n \models [k_m, k_{m+1}] \leq F_t. \]
(Note \(p_t \models \forall i \geq m + 1 \tau(i) > k_{m+1})).

Let \(p_{n+1}(0) = (p_n(0) - p_n(0) \land \forall(q_t(0) : t \text{ immediately below } s \text{ in } p_n(0)). \)
Let $p_{n+1} (1, \omega_2)$ be a term denoting

\[ q_t \uparrow (1, \omega_2) \text{ if } q_t(0) \]
\[ p_n \uparrow (1, \omega_2) \text{ if } p_n(0) - \{t : s \leq t\} \]

so $p_{n+1} \leq p_n$.

Now let $p$ be the fusion of the sequence of $p_n$ (see Lemma 5 [L]).

Then for each $t \in \hat{p}(0)$ if $t = <k_0, \ldots, k_m, k_{m+1}> \land t \geq

\text{stem } \hat{p}(0)$ then

$< \hat{p}(0) \uparrow \hat{p} \uparrow (1, \omega_2) > \models " \forall \tau \in [k_n, k_{n+1}] \subseteq F_t".$

For $t \in \hat{p}(0) \land t \in \text{stem } \hat{p}(0)$ let $F_t = k_{m+1}$.

Applying Lemma 1 obtain $T_0, T_1 \models \hat{p}(0)$ $Z_0, Z_1, Z_0 \land Z_1$ finite

$< T_i \uparrow \hat{p} \uparrow (1, \omega_2) > \models " \forall \tau \in Z_i" \text{ i = 0, 1. }$ 

Proof of the Theorem:

Suppose $M[G_{\omega_2}] \models "U \text{ is a semi q-pt.}"$

Applying an argument of Kunen's we get $\alpha < \omega_2$

$U \uparrow M[G_\alpha] \in M[G_\alpha]$.

$(M[G_\beta] \models "\text{CH}" \text{ all } \beta < \omega_2$ so construct using $\omega_2$-c.c.

$\alpha_\lambda < \omega_2, \lambda < \omega_1$ so that $\forall x \in M[G_\alpha^\lambda] \land 2^\omega, P_{\alpha_\lambda+1}$ decides

"$x \in U". \text{ Let } \alpha = \sup \alpha_\lambda$. Note

$M[G_\alpha] \land 2^\omega = \bigcup_{\beta < \alpha} M[G_\beta] \land 2^\omega$ since $\alpha_\lambda^\lambda$ is not collapsed.)

By Lemma 11 [L] we may assume $U \land M \in M$. But Lemma 2 clearly implies that for any $V$ ult. in $M, M[G_{\omega_2}] \models "\text{no extension of } V \text{ is a q-pt}".$
Remarks:

1) A similar argument shows that in model gotten by $\omega_2$ iteration of Mathias forcing with countable support there are no semi-q-pt's.

2) In [M] Mathias shows $(\omega + (\omega)^\omega) \rightarrow$ (no rare filters or non-principal ultrafilters).

3) In neither the Laver or Mathias models are there small dominant families so by Kettenon [Ke] $\exists$ p-pt's. Also it is easily shown no ultrafilter is generated by fewer than $\aleph_2$ sets.

Conjecture: Borel conjecture $\iff \exists$ semi q-point in $\mathfrak{B}N-N$. 
REFERENCES


§1. Universal clopen sets, Wadge degrees, and \(\omega\)-Boolean operations

Given \(B \subseteq \omega^\omega \times \omega^\omega\) define \(B_x = \{y: (x,y) \in B\}\). \(B\) is said to be universal for clopen sets \((\Delta^0_1)\) iff

\[\forall x \ B_x \in \Delta^0_1 \text{ and } \forall A \in \Delta^0_1 \exists x(A = B_x).\]

What is the simplest \(B\) universal for clopen sets? The reader is obliged to guess before reading on. (For example good choices seem like: open, difference of closed sets, \(F_\delta(\Sigma^0_2)\), etc.)

The complement of a set universal for clopen sets is also.

Here is a \(\Pi^1_1\) definition. Let \(A \subseteq \omega^\omega \times \omega^\omega\) be open and universal for open sets.

\((x,y) \in B\) iff \(\forall z((z \in A(x)_0 \iff z \notin A(x)_1) \land y \in A(x)_0)\)

\(((x)_i\) recursive uncodings).

**Theorem 1.** On the other hand no Borel set is universal for clopen sets.

**Proof.**

For \(C \subseteq \omega^\omega\) and \(s \in \omega^{<\omega}\) \((s \in Fr(C)\) iff \(\exists y,z \ y,z \geq s(y \in C \land z \notin C)\).

\(A = \{T \subseteq \omega^\omega: \exists x \forall s \in T (s \in Fr(B_x)) \text{ and } T \text{ closed under subseq.}\}\). If \(B\) were Borel and universal for clopen sets then \(A\) would be a \(\Xi^1_2\) set of well founded trees of arbitrarily high rank, contradicting the boundedness theorem. \(\blacksquare\)
I don't know the answer for sets universal for $\Delta^0_\alpha$ sets

$2 < \alpha < \omega_1$. Harrington has proved Theorem 1 for $\Delta^0_\omega$ sets.

Similar questions are settled by C.A. Rogers [1] and Kechris-Martin [2].

Define: $A \preceq_w B$ for $A \subseteq X, B \subseteq Y, X, Y$ topological spaces (Wadge [3]) iff $\exists f: X \to Y$ continuous and $f^{-1}(B) = A$.

Given $T \subseteq 2^\omega$ (truth table) define the $\omega$-boolean operation $\Gamma_T: (P(X))^{<\omega} \to P(X)$ for any $X$ by

$$(x \in \Gamma_T((A_n: n < \omega)) \iff \{n: x \in A_n\} \in T)$$

where we identify $2^\omega$ with $P(\omega)$.

Some examples of $\omega$-boolean operations are countable union, operation $\mathcal{R}$, $R$-operations of Kolmogorov [5], and the Borel game operations of Burgess [6].

Define $\mathcal{C}_T = \{A \subseteq 2^\omega: \exists(B_n: n < \omega)$ each $B_n$ clopen and $\Gamma_T(B_n: n < \omega) = A\}.$

**Theorem 2.** For any $T \subseteq 2^\omega \mathcal{C}_T = \{A \subseteq 2^\omega: A \preceq_w T\}$.

**Proof.**

(2)

Define $A_n \subseteq 2^\omega$ by $a \in A_n$ iff $a(n) = 1$. The $A_n$'s are clopen. Suppose $B \preceq_w T$ via continuous function $f: 2^\omega \to 2^\omega$ and $B_n = f^{-1}(A_n)$. Then each $B_n$ is clopen and

$$(\beta \in \Gamma_T < B_n: n < \omega >) \leftrightarrow (\{n: \beta \in B_n\} \in T) \leftrightarrow (\{n: f(\beta) \in A_n\} \in T) \leftrightarrow (f(\beta) \in T) \leftrightarrow (\beta \in B),$$

hence $B \in \sigma_T$. 


Let \((B_n : n < \omega)\) be a \(\omega\) sequence of clopen sets. We must use the following fact due to Wadge:

Fact: for \(A, B \subseteq 2^\omega, A \leq \omega B\) iff player II has a winning strategy in \(G(A, B)\). \(G(A, B)\) is the game where player I and player II alternatingly write down 0 or 1 creating two maps \(\alpha : \omega \to 2\) and \(\beta : \omega \to 2\) respectively. On his moves player II may elect to pass but he must play infinitely often if I does. Player II wins a particular play \((\alpha, \beta)\) iff \((\alpha \in A \text{ iff } \beta \in B)\).

Claim: II wins \(G(\Gamma_T(B_n : n < \omega), T)\).

Proof.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha(0))</td>
<td>(\beta(0))</td>
</tr>
<tr>
<td>(\alpha(1))</td>
<td>(\beta(1))</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>(\alpha(n))</td>
<td>(\beta(n))</td>
</tr>
</tbody>
</table>

II waits until either \([\alpha \upharpoonright n] \subseteq B_\varnothing\) or \([\alpha \upharpoonright n] \cdot B_\varnothing = \emptyset\) then plays 1 or 0 accordingly. Since \(B_\varnothing\) is clopen he will not have to wait indefinitely. If he continues to play in this fashion he produces a play \(\beta \in 2^\omega\) so that

\(\alpha \in B_n\) iff \(\beta(n) = 1\), thus

\(\alpha \in \Gamma_T \triangleright B_n : n < \omega \iff \{n : \alpha \not\in B_n\} \in T \iff \beta \in T\)

This theorem was proved by myself and Lon Radon. Other similar questions for \(\omega^\omega\) in place of \(2^\omega\) and open in place of clopen are answered by Steel [8] and Van Wesep [9].
The next question I consider is whether or not there is a natural hierarchy for the \( \Delta^1_2 \) subsets of \( \omega^\omega \). The only results known are negative, for example Moschovakis [10].

Note that if \( \{ T \} \cup \{ A_n : n < \omega \} \leq \Delta^1_2 \) then

\[ \Gamma_T(A_n : n < \omega) \leq \Delta^1_2. \]

In general for \( \mathcal{C} \leq \mathcal{P}(\omega^\omega) \) let \( \mathcal{C}^* \) be the least containing \( \mathcal{C} \) and if \( \{ T \} \cup \{ A_n : n < \omega \} \leq \mathcal{C}^* \) then \( L_T(A_n : n < \omega) \in \mathcal{C}^* \). (Note \( ((\Delta^0_1)^*) = \Delta^0_1, (\Sigma^0_1)^* = \Delta^1_1 \)).

Using the method of Kunugie [11] we prove:

**Theorem 3.** Suppose \( \mathcal{C} = \{ A : A \subseteq B \} \) where \( B \in \Delta^1_2 \) then

\[ \exists \mathcal{C} \in \Delta^1_2 \mathcal{C}^* \leq \{ A : A \subseteq \mathcal{C} \}. \]

**pf**

Define \( U \subseteq \omega^\omega \times \omega^\omega \) by \( (x,y) \in U \) iff \( f_y(x) \in B \).

\( f_y : \omega^\omega \to \omega^\omega \) is the continuous function \( \Delta^1_1 \) coded by \( y \).

Then \( U \) is \( \Delta^1_1 \), and \( \forall A \in \mathcal{C} \exists y \in \omega^\omega U_y = A \).

Define \( x = (T,f) \) is a code iff \( T \leq \omega^{<\omega} \) is a well-founded normal tree and \( f : \{ s \in T : |s|_T = 0 \} \to \omega^\omega \).

Define \( C_x \) for \( \sigma \in T \) as follows:

\[ |\sigma|_T = 0 \text{ then } C_x = U_f(\sigma) \]
\[ |\sigma|_T > 0 \text{ then } C_x^\sigma = \Gamma_{C_x} (C_x^{\sigma_i+1} : i < \omega). \]

Define \( P(x,y) \) iff "\( x \) is a code and \( y \in C_x^\phi \". Clearly \( \mathcal{C}^* \subseteq \{ A : A \subseteq \omega P \} \), and it is not hard to show that \( P \) is \( \Delta^1_2 \).
§2 Wadge degrees of orbits

Let $\mathcal{A}$ be a countable structure. Define

$$[\mathcal{A}] = \{(\omega, R_n : n < \omega) : \mathcal{A} \text{ is isomorphic to } (\omega, R_n : n < \omega)\}.$$  

$[\mathcal{A}]$ is called the orbit of $\mathcal{A}$. Scott's Theorem [15] says that for any countable $[\mathcal{A}]$ is Borel. Recall $\delta$ the metric defined in Part II. For any $a \in A$ let $\mathcal{A}(n,a) = \{b \in A : \delta(a,b) < n\}$. $\mathcal{A}$ has finite valency ([14]) iff for any $a \in A$ and $n < \omega$ $\mathcal{A}(n,a)$ is finite.
Theorem 4. For every $\mathcal{O}$ finite valency, countable

$[\mathcal{O}] \in \Pi^0_\omega$

Proof.

For $a \in A$ let

$P_a(v) = \bigwedge_{n<\omega} [\exists x_1, \ldots, x_j \theta^a_n(v, x_1, \ldots, x_j) \land \exists x_1, \ldots, x_j \exists y (\theta^a_n(v, x_1, \ldots, x_j) \land \delta(v, y) \leq n \land \forall v \neq y \land \bigwedge_i x_i \neq y)]$ where $\mathcal{O}(a, n) = \{a, b_1, \ldots, b_j\}$ and

$\theta^a_n(a, b_1, \ldots, b_j)$ is the conjunction of all atomic sentences and negations of atomic sentences involving $a, b_1, \ldots, b_j$ and some $R_m$ for $m < n$ or equality.

Lemma 1. $\forall \mathcal{P}$ countable (not necessarily of finite valency)

$\forall b \in |\mathcal{P}|, \mathcal{P} \models P_a(b) \iff \mathcal{O}(\omega, a) \equiv \mathcal{O}(\omega, b)$.

Proof.

Clearly $\mathcal{P} \models P_a(b)$ implies there are isomorphisms

$F_n: <\mathcal{O}(n, a), R_m>_{m<n} \rightarrow <\mathcal{O}(n, b), R_m>_{m<n}$ each sending $a$ to $b$.

Define a back and forth property $\mathcal{I}$ by:

$((a, \mathcal{B}) \epsilon \mathcal{I}) \iff$ infinitely many $n, F_n(\bar{a}) = \mathcal{B}$.

$(a, b) \epsilon \mathcal{I}$ so $\mathcal{I}$ is not empty. Suppose $(\bar{a}, \mathcal{B}) \epsilon \mathcal{I} \land c \epsilon \mathcal{O}(\omega, a)$. Choose $N < \omega$ so that $\mathcal{O}(N, a)$ contains $(\bar{a}, c)$, then since $|\mathcal{O}(N, a)|, |\mathcal{O}(N, b)|$ are finite

$\exists d \epsilon |\mathcal{O}(N, b)|$ such that infinitely many of the $F_n$ sending $\bar{a}$ into $\mathcal{B}$ send $c$ into $d$. (Same argument for other direction of the back and forth.)
Choose \(a_n\) for \(n < N \leq \omega\) so that \(\forall \varepsilon\) a component (maximally connected) of \(\sigma\). \(\exists n < N \varepsilon \sigma (\omega, a_n)\).

Define \(g: N \rightarrow \omega + 1, g(n) = \text{number of components of } \sigma\) isomorphic to \(\sigma (\omega, a_n)\).

Let \(\theta(v, w) = \bigwedge_{n<\omega} \delta(v, \omega) \geq n\).

Let \(\psi = (\bigvee x \bigwedge_{a \in \aleph_1} P_a(x)) \land (\bigwedge_{n<\omega} \bigwedge_{m<g(n)} \exists x_0, \ldots, x_m (i \neq j \theta(x_i, x_j) \land \bigwedge_{i \leq n} P_{a_n}(x_i)) \land (\bigwedge_{i \leq g(n)} P_{a_n}(x_i))\).

Since \(\theta\) is \(\Pi^0_1\) and \(P_a(v)\) is \(\Pi^0_1\) we have that \(\psi\) is \(\Pi^0_2\). Theorem 4

We show this is best possible:

**Theorem 5.** If \(\sigma_1 = (\omega, R) R \leq \omega^2\) is the graph of the 1-1 function whose components consist of infinitely many copies of \((\omega, Sc)\) and infinitely many copies of \((\mathbb{Z}, Sc)\) \((Sc(x) = x + 1)\) then \([\sigma_1] \not\in \Pi^0_4\).
Play the following game of Solitary: On the $n^{th}$ move you are presented with the $n^{th}$ row of zero's and one's (seemingly at random) in $\omega$ columns $C_m$ for $m < \omega$. You (eventually to write down a structure $< \omega, R >$, $R \subseteq \omega^2$) write down an extension $\sigma_n$ of $\sigma_{n-1} \subseteq \sigma_n$ with universe contained in $\omega$. Let $\sigma = \bigcup_{n<\omega} \sigma_n$. In order to win this game you must arrange that the universe of $\sigma$ is $\omega$ and either

$$\sigma \cong \sigma_0 = \aleph_0^\omega$$-copies of $< \omega, Sc >$

or

$$\sigma \cong \sigma_1 = 1 \text{- copy of } < \mathbb{Z}, Sc > + \sigma_0 \ .$$

In addition you must guarantee:

$$\sigma \cong \sigma_1 \text{ iff (one of the columns } C_n \text{ has infinitely many one's in it.)}$$

It might be easier for the reader to find his or her own argument. Any finite structure isomorphic to $< n, Sc >$ for $n < \omega$ will be called a string. Here is a rough description of a winning strategy:
After \( n \) moves of the game, \( \mathcal{A}_n \) will consist of finitely many strings labeled \( C_0, C_m \) and the rest of the strings will all be labeled \( G \) (for garbage). The first thing we do is push each string forward, i.e. given:

\[
\begin{array}{cccc}
1 & 2 & 3 & 1 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

we add another element making \( 1 \rightarrow 0 \rightarrow 0 \rightarrow 0 \), and we also create a new string \( 0 \rightarrow 0 \) and label it \( G \).

Next we look to see if a 1 appears in the \( n \)th row in any column. If none appears we're done with this move. Otherwise let \( k_0 < \omega \) be the least \( k \) with 1 appearing in the \( k \)th column \( C_k \) and \( n \)th row. We move the string labeled \( C_{k_0} \) back: i.e. \( C_{k_0} = 1 \rightarrow 2 \rightarrow 3 \) becomes \( 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \). And we take all strings labeled \( C_k \) for \( k > k_0 \) and relabel them \( G \) (injured priorities). This is a winning strategy because if none of the columns have infinitely many one's in them, then no copy of \( \langle Z, Sc \rangle \) is ever made. So \( \mathcal{A} = \mathcal{A}_0 \).

If \( k_0 \) is the least \( k \) such that \( C_k \) has infinitely many one's in it then at some stage \( n_0 < \omega \) none of the columns \( C_k \) for \( k < k_0 \) ever get a one in them. After this point the string labeled \( C_{k_0} \) is made into \( \langle Z, Sc \rangle \) and each \( C_k \) makes into \( \langle \omega, Sc \rangle \) for \( k < k_0 \) as do things in \( G \). So \( \mathcal{A} = \mathcal{A}_1 \).

If \( H \in \Pi^0_1 \) then \( \exists C : 2^\omega \rightarrow \mathcal{P}(\omega) \) continuous so that \( \forall x (x \in H \iff C(x) \text{ infinite}) \). Hence for any \( G \in \Sigma^0_3 \)

\( \exists \langle G_n : n < \omega \rangle \) cont. so that \( (x \in G \iff \exists n \ G_n(x) \text{ infinite}) \).
Let \( \bigcap_{n \in \omega} G_n \) be any \( \Pi^0_1 \) set \( G_n \in \Sigma^0_3 \) and \( G_0 \supseteq G_1 \supseteq G_2 \ldots \). Then using above game we have maps \( f_n : 2^\omega \to 2^{\omega \times \omega} \) continuous so that \((x \in G_n) \mapsto (f_n(x) \neq \mathcal{O}_n) \). So easily we have a continuous map \( g : 2^\omega \to 2^{\omega \times \omega} \forall x (x \in G_n) \mapsto (g(x) \neq \mathcal{O}) \), showing \([\mathcal{O}]\) is complete \( \Pi^0_1 \).

Remark: Given \( \langle v, E \rangle \) a countable graph \((E \subseteq [v]^2 = \{(a, b) \subseteq v : a \neq b\})\) define \( \mathcal{O} = \langle v, \omega E, \varepsilon \rangle \) by \((a \in b) \mapsto (a \in V \land b \in E \land b = \{a_0, a_1\} \land (a = a_0 \lor a = a_1))\). Then \( \mathcal{E} \) on the universe of \( \mathcal{O} \) is a relation with disjoint domain and range (hence a partial order). Furthermore \( \omega(\text{Th} (\mathcal{O})) = \omega(\text{Th} (\langle v, E \rangle)) \). Theorem 4 shows that \( \mathcal{O} \) has \( \leq \aleph_0 \) or \( 2^{\aleph_0} \) non-isomorphic substructures. The proof is as follows.

Case 1. There are infinitely many \( x \in V \) of infinite valency \(||\{y : x Ey\}|| = \aleph_0\). Build a distinct sequence \( x_n \in V, Y_n \subseteq V \) infinite for \( n < \omega \) so that \( \forall n \forall y \in Y_n \langle \{x_n, y\} \in E \rangle \) and \( \forall n \neq m \langle Y_n \cap Y_m = \emptyset \rangle \). Looking at substructures of \( \mathcal{O} \) allows us in effect not only to drop out vertices (elements of \( v \)) but also edges (elements of \( E \)). Hence we may "drop" all edges except those connecting each \( x_n \) to the elements of \( Y_n \) and easily show \( \mathcal{O} \) has \( 2^{\aleph_0} \) non-isomorphic substructures.
Case 2. There are only finitely many $x \in v$ of infinite valency. By an easy generalization of Theorem 4 the orbit of every substructure of $C$ is $\Pi^0_\omega$, hence we conclude that the class of substructures of $C$ (note that it is $\text{PC}(L_{\omega_1\omega})$) obeys Vaught's conjecture.

Next we characterize the Wadge degrees of well orderings.

Theorem 6. If $\beta = \lambda + m$ where $\lambda$ is a limit ordinal and $m < \omega$ and $\gamma = \omega^\beta \cdot n + \delta$ where $n < \omega$ and $\delta < \omega^\beta$, then if $n = 1$ then $[(\gamma, <)]$ is $\Pi^0_{\lambda+2m+1}$ properly and if $n > 1$ then $[(\gamma, <)]$ is $2 - \Pi^0_{\lambda+2m+1}$ properly.

Proof.

The computation of the upper bound on the complexity will be left to the reader. Now we show that the orbits are properly of the given complexity.

Define $C \equiv_{\alpha} \mathcal{X}$ iff $C$ and $\mathcal{X}$ model the same $\Pi^0_{\alpha}$ sentences. $C \preceq_{\alpha} \mathcal{X}$ is defined similarly.

Lemma 2. If $\beta = \lambda + n$ where $\lambda$ is a limit ordinal and $n < \omega$, then $(\omega^\beta, <) \preceq_{\lambda+2n}(\omega^\beta \cdot \delta, <)$ for any $\delta > 0$. 
Proof.

This is essentially what Ehrenfeucht shows in Theorem 12 of [12], except he does not go into transfinite levels, and in his game $H_n$, player I gets to choose which model he
wants to play in each turn.

Instead we should play the following game $K_\alpha(\sigma, \nu)$.

Player I begins by picking $\sigma$ or $\nu$ on his 1st move and playing a finite sequence from the model he picks, from then on he must alternate between $\sigma$ and $\nu$, on each turn he plays an ordinal $\beta_n$ with $\beta_{n+1} < \beta_n$. The game is over when I plays zero.

It can be shown that if Player II has winning strategy in $K_\alpha(\sigma, \nu)$, then $\sigma \equiv_\alpha \lambda^\mu$.

Now consider $\beta > 0$, $0 < n < \omega$, $\delta < \omega^\beta$, $\beta = \lambda + m$. By the Lemma

$\langle \omega^\beta + \delta, \delta \rangle \leq \lambda + 2m \leq \omega^\beta \cdot n + \delta, \delta \rangle \leq \omega^\beta \cdot (n+1) + \delta, \delta \rangle$

thus for any $\Sigma^0_{\lambda+2m+1}$ sentence $\theta$:

* if $\sigma_0 \models \theta$ then $\sigma_1 \models \theta$

** if $\sigma_1 \models \theta$ then $\sigma_2 \models \theta$

thus $\langle \omega^\beta + \delta, \delta \rangle$ is not $\Sigma^0_{\lambda+2m+1}$ by *

Suppose

$\langle \omega^\beta \cdot n + \delta, \delta \rangle$ were $\text{co}(\Pi^0_{\lambda+2m+1})$,

then $\langle \omega^\beta \cdot n + \delta, \delta \rangle$ would be union of a $\Pi^0_{\lambda+2m+1}$ and $\Sigma^0_{\lambda+2m+1}$ hence would be $\Pi^0_{\lambda+2m+1}$ contradicting * or $\Pi^0_{\lambda+2m+1}$ contradicting **.
Lemma 3. If $\alpha \neq \omega$, $\alpha \neq \omega$ then orbit of $\alpha$ is not $\Sigma^0_{\omega+1}$ and orbit of $\omega$ is not $\Pi^0_{\omega+1}$. Now we give some examples of other orbits.

Define $i, j \leq \omega$ $\alpha, \omega$ $\mathfrak{P}$-structures then $i \cdot \alpha + j \cdot \omega$ is the following $\mathfrak{P} \cup \{\sim\}$ structure.

$|i \cdot \alpha + j \cdot \omega| = i$ copies of $|\alpha| \cup j$ copies of $|\omega|$

$x \sim y$ iff $x, y$ are in same copy of $|\alpha|$ or $|\omega|$

$R\overline{x}$ iff $\overline{x}$ are in same copy of $|\alpha|$ or $|\omega|$ and $R\overline{x}$ holds there.

Lemma 4. If $\alpha \leq \omega$ then $i \cdot \alpha + \omega \cdot \omega \not\in_{\alpha+1} (i+1) \cdot \alpha + \omega \cdot \omega$

Proof.

Easy playing game.

Lemma 5. If $\alpha \geq 2$ $\alpha^+ \leq \omega$ then $\omega \cdot \omega$ is $\Pi^0_{\omega}$ and $1 \cdot \alpha + \omega \cdot \omega$ is $2 \cdot \Pi^0_{\alpha}$.

Proof.

Scott sent for $\alpha$

Scott sent for $\omega$

For any $x, \psi$, $\psi^x$ the formula obtained by relativizing the quantifier of $\psi$ to $\{y: y \sim x\}$.

Let $\psi_0 \equiv (a) \forall \mathfrak{P} \exists x (R\overline{x} \rightarrow \exists i, j x_i \sim x_j) \wedge$

(b) $\sim$ equivalence relation $\wedge$

(c) $\forall j x_i \neq x_j \wedge$

(d) $\forall x \theta^x_1$. 
$\psi_0$ is a Scott sentence for $\omega \cdot \mathcal{L}$.

Let $\psi_1 \equiv (a) \land (b) \land (c) \land$

$(e) \forall x(\theta^x_0 \lor \theta^x_1) \land$

$(f) \forall x, y x \neq y \rightarrow (\theta^x_0 \land \theta^y_0) \land$

$(g) \exists x \theta^x_0$.

$\psi$ is $2 \cdot \Pi^0_\omega$ Scott sentence for $1 \cdot \alpha + \omega \cdot \mathcal{L}$.  

If $\mathfrak{a}, \mathcal{L}$ p-structures $L, L'$ linear orders, then define $\mathfrak{a} \cdot L + \mathcal{L}'$ the $p \cup \{\leq\}$-structure as follows.

Let $\mathfrak{a}_l, \mathcal{L}'_k$ be copies of $\mathfrak{a}, \mathcal{L}$ for each $l \in L, k \in L'$

$|\mathfrak{a} \cdot L + \mathcal{L}'| = \bigcup_{l \in L} |\mathfrak{a}_l| \cup_{k \in L'} |\mathcal{L}'_k|$

$x \leq y$ iff $[\exists l \exists k (x, y \in \mathfrak{a}_l) \lor (x, y \in \mathcal{L}'_k) \lor (x \in \mathfrak{a}_l \land y \in \mathcal{L}'_k) \lor (l \leq k \land x \in \mathfrak{a}_l \land y \in \mathcal{L}'_k)]$.

R$\bar{x}$ $\leftrightarrow [\bar{x}$ in one copy of $\mathfrak{a}$ or $\mathcal{L}$ and $\bar{R}$ holds there].

**Lemma 6.** If $\mathfrak{a} \in \mathcal{L}$ then $\mathfrak{a} \cdot \eta + \mathfrak{a} \cdot \eta \in \mathcal{L}^{(1+\eta)}$.

**Proof.**

Easy using games--the extra copies of $\mathfrak{a}$ on left correspond to some $\mathfrak{a}_s$ $s \in \eta$ on left.  

**Lemma 7** ($\alpha \geq 2$). If $[\mathfrak{a}], [\mathcal{L}] \in \Delta_\alpha^\delta$ then

$[\mathcal{L} \cdot \eta + \mathfrak{a} (1+\eta)] \in \Sigma^\delta_{\alpha+1}$. 

Proof.

Define \( x \sim y \) iff \( x \leq y \land y \leq x \). Let

\( \theta^0 \) Scott sentence for \( \sigma^1 \) and

\( \theta^1 \) Scott sentence for \( \mu^0 \). Then the conjunction of

(a) \( \leq \) has order type \( \eta \)
(b) \( \bigwedge_{p \in \mathbb{P}} \mathfrak{M}_R x \to \bigwedge_{i \in \mu} x_i \sim x_j \)
(c) \( \forall x \forall y (\theta^x_0 \land \theta^x_1 \to x > y) \)
(d) \( \forall x (\theta^x_0 \lor \theta^x_1) \)
(e) \( \exists x \theta^x_0 \land \forall y < x \theta^y_1 \)

is a \( \Sigma^0_{\alpha+2} \) Scott sentence for \( \mu^0 \land \sigma^1 \cdot (1+\eta) \).

Theorem 7. For each \( \alpha \), \( 0 < \alpha < \mathfrak{K}_1 \) there are orbits which
are properly:

\( \Pi^0_{\alpha+1}, \Pi^0_{\alpha}, \Pi^0_{2-\alpha+1}, \Pi^0_{2-\alpha+1} \), and \( \Sigma^0_{\alpha+2} \).

Proof.

The ordinals give examples of \( \Pi^0_{\lambda+2n+1} \) or \( \Pi^0_{\lambda+2n+1} \)
orbits for \((\lambda > 0 \text{ limit } 0 \leq n < \omega)\) or

\((\lambda = 0 \Rightarrow 1 \leq n < \omega)\). For \( \lambda \) a limit \( \Rightarrow \), choose \( \alpha_n \uparrow \omega^\lambda \).

It is easy to see that the orbit

\([\omega^\lambda, <, p>]\) where \( p = \{ \alpha_n : n < \omega \} \) is \( \Pi^0_{\lambda} \) and not \( \Sigma^0_{\lambda} \).

Now let \( \sigma^1 = [\omega^{\alpha+n}, <, \lambda + n, \omega, \omega^\alpha] = \mu^0 \).

By Lemma 4 \( (\omega \cdot \mu^0) \to \lambda + 2n + 1 \cdot (\omega^\alpha + \omega \cdot \omega^\alpha) \)

\( \to \lambda + 2n + 1 \cdot (\omega^\alpha + \omega^\alpha) \).
Since \( \sigma_1, \nu \) have \( \Delta_0^{\omega \cdot 2n+2} \) orbits by Lemma 5, 
\( \omega \cdot \nu \) is \( \Pi_0^{\omega \cdot 2n+2} \) and \( 1 \cdot \sigma_1 + \omega \cdot \nu \) is \( 2 - \Pi_0^{\omega \cdot 2n+2} \).
They are properly so by Lemma 2.

Thus we have examples of proper \( \Pi^0_{\omega \cdot 2n+2} \), \( 2 - \Pi^0_{\omega \cdot 2n+2} \), \( \alpha \geq 3 \) orbits.

In fact \( \forall \alpha, 2 < \alpha < \omega \) we have structures \( \sigma_1 < \omega \) such that 
\( \text{orb}(\sigma_1), \text{orb}(\nu) \) are \( \Delta_0^{\omega \cdot \alpha + 2} \). By Lemma 6
\( \nu^* \cdot \eta + \sigma_1 \cdot \eta < \omega \cdot \alpha + 2 \cdot \nu^* \cdot \eta + \sigma_1 (1+n) \).

Hence by Lemma 5 \( \nu^* \cdot \eta + \sigma_1 (1+n) \) is not \( \Pi^0_{\omega \cdot \alpha + 3} \).

By Lemma 7 
\[ [\nu^* \cdot \eta + \sigma_1 (1+n)] \text{ is } \Sigma^0_{\omega \cdot \alpha + 3} \).

Now let \( \sigma_1, \nu \) be the following structures in one relation symbol \( \sim \).

\( \sim \) equivalence relation one equivalence class.

\( \sim \) equivalence relation two equivalence classes, one of which has size \( 1 \).

It is easy to see that \( [\sigma_1] \) is \( \Pi_1^0 \),
\( \Lambda [\nu] \) is \( 2 - \Pi_1^0 \).

Since \( \sigma_1 \in \sigma_1, \nu \) we have \( [\nu^* \cdot \eta + \sigma_1 (1+n)] \) is \( \Sigma_1^0 \).

Now let \( \sigma_1 = < \mathbb{Z}, Sc >, \ 
u = < \mathbb{Z} + \mathbb{Z}, Sc >, \)
then \( \sigma_1 \) is \( \Pi_2^0 \cdot \nu \) and \( \sigma_1 \notin \nu \) so by above \( \nu^* \cdot \eta + \sigma_1 (1+n) \) is \( \Sigma_2^0 \).

This gives examples of all orbits promised except for
\( \Sigma_2^\lambda \cdot \nu \) limit \( \lambda > 0 \); which we now provide: (keep in mind to \( 0 \)).

the structure \( < \emptyset, C_n, < > \) where \( C_n \) are strictly increasing.

Suppose we have \( p \)-structures \( \sigma_1, \sigma_\nu, \) and \( \sigma, \) then \( \nu \) is the following \( p \cup \{ < > \} \)-structure.
\[ |\mathcal{P}| = \{ \langle r, a \rangle : r \in \mathcal{Q}, a \in |\sigma_n| \text{ if } r = c_n \text{ for some } n < \omega \text{ otherwise} \} \].

\[(r, a, r < (s, b)) \leftrightarrow (r \leq s) \].

\[ R^x = (\exists r. x = \langle r, a \rangle, \langle r, a_2 \rangle, \ldots, \langle r, a_n \rangle) \text{ and } \]

\[ Ra_1, \ldots, a_n \text{ holds in appropriate structure} \].

\[ \hat{\mathcal{N}}^r = \mathcal{N} - \{ \langle 0, a \rangle : a \in |\sigma| \}. \]

**Lemma 8.** Suppose \( \alpha_n \models \sigma_n \leq_{\alpha_n} \sigma_\lambda \) then \( \hat{\mathcal{N}}^r \models \sigma_\lambda \).  

**Proof.**

Easy using game criterion.

**Lemma 9.** Orbit of \( \mathcal{N} \) is \( \Sigma^0_{\lambda+2} \).  

**Proof.**

Just write it all down.

**Note:** If \( \omega_n + \lambda \) then \( (\omega^\lambda_n, <) \not\models (\omega^\lambda, <) \).

This concludes proof of Theorem 7. \[ \square \]

**Remark:** An immediate corollary of D. Miller's invariant difference hierarchy theorem [13] is that if \( [\sigma] \in \Delta^0_{\alpha+1} \) then there are invariant \( \Pi^0_\alpha \) sets \( A \) and \( B \) so that \( [\sigma] = A \land \neg B \). Also a theorem of Vaught [19] says that a \( \Pi^0_\alpha \) set \( B \) is invariant iff \( B \) is the set of countable models of some \( \Pi^0_\alpha \) sentence of \( L_{\omega_1 \omega} \). Thus if \( [\sigma] \subseteq B \) where \( B \) is an invariant.
$\Sigma_0^0$ set then $[\alpha] \leq B' \leq B$ where $B'$ is an invariant
$\Pi_0^0$ set some $n < \omega$. The following diagram summerizes
the content of these remarks and Theorems 7 and 8. The only
open question is: Are there any $\Sigma_{\lambda+1}^0$ orbits for $\lambda > 0$
a limit?

Yes $\Pi_{-1}^0$ $\Sigma_{-1}^0$ No

Yes $\Pi_{-2}^0$ $\Sigma_{-2}^0$ No

Yes $\Pi_{-3}^0$ $\Sigma_{-3}^0$ Yes

... ...

Yes $\Pi_{-\omega}^0$ $\Sigma_{-\omega}^0$ No

Yes $\Pi_{-\omega+1}^0$ $\Sigma_{-\omega+1}^0$ ?

Yes $\Pi_{-\omega+2}^0$ $\Sigma_{-\omega+2}^0$ Yes

In [13] D. Miller proves that in the topology generated
by first order formulas there are no $\Sigma_2^0$ orbits. Next
we show in the usual topology that such orbits are
impossible.
Theorem 8. Proper $\Sigma_0^0$ orbits are impossible.

Proof.

Suppose $\mathcal{A} = \langle A, R \rangle$, $|A| = \aleph_0$, $R$ countable similarity
type containing only relation symbols. Suppose

$$\theta_0 = \exists x_0, \ldots, x_{n-1} \land \forall m < \omega \psi_m(x_0, \ldots, x_{n-1})$$

where $\psi_m(x) \in \Pi_1^0$ formula of 1st order logic.

Suppose

$$\forall \mathcal{M} \models \theta_0 \iff \mathcal{M} = \mathcal{A} \implies \mathcal{M} \models \theta_0.$$ 

Lemma 10. $\mathcal{A}$ is $\omega$-saturated (in fact $\mathcal{M}(\mathcal{A})$ is

$\aleph_0$-categorized).

Proof.

$\mathcal{A}$ is weakly saturated. To see this let $\Sigma$ be a type
consistent with $\mathcal{M}(\mathcal{A})$. Let $\mathcal{N} \models \mathcal{A}$ be countable
realizing $\Sigma$. Since $\mathcal{N} \models \theta_0$, $\mathcal{N} = \mathcal{A}$. So
$\mathcal{A}$ is weakly saturated. Thus $\mathcal{M}(\mathcal{A})$ has only countably many $n$-types
each $n$. So there exists $\mathcal{N}$ countable $\omega$-saturated
model of $\mathcal{M}(\mathcal{A})$. Since $\mathcal{A} \times \mathcal{N} \models \theta_0$ so

$\mathcal{N} = \mathcal{A}$. 

Define $\theta(x_0, \ldots, x_{n-1}) = \bigwedge_{j \neq i} (x_i \neq x_j) \land$

$$\bigwedge_{f \in \mathrm{n!}} \bigwedge_{m < \omega} \psi_m(x_f(0) \ldots x_f(n-1))$$

where $\mathrm{n!}$ is the symmetric group on $n$.

Thus $\theta \mathcal{A} = \{ x \in [A]^n : \mathcal{A} \models \theta(x) \}$ partitions the $n$-element
subset of $A ( [A]^n )$. 


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By Ramsey's theorem \( \exists v \subseteq A \) inf. so that 
\[ [v]^n \subseteq \theta^\sigma \] or \([v]^n \subseteq [A]^n - \theta^\sigma \] (\( \equiv \sim \theta^\sigma \)).

If the first happens then we have \( \langle v, R \rangle \models \forall \vec{x} \theta(\vec{x}) \) and hence by * \( \forall \vec{x} \theta(\vec{x}) \) is a \( \Pi^0_1 \) Scott sentence for \( \sigma^\tau \).

So we assume \([v]^n \subseteq \sim \theta^\sigma\).

Choose \( B \in \theta^\sigma \) and throw out of \( v \) any part of \( B \).

By repeatedly applying Ramsey's theorem we obtain \( \hat{v} \subseteq v \) infinite so that

\[ \forall F \subseteq B \text{ either (a) } \forall G \in [v]^n-\|F\| \text{ F } \cup \text{ G } \subseteq \theta^\sigma \]

or (b) \( \forall G \in \hat{v}^n-\|F\| \text{ F } \cup \text{ G } \notin \theta \)

Choose \( F \subseteq B \) of minimal cardinality so that (a) happens (it always exists since \( F = B \) will do).

Let \( F = A_0 \), \( \|A_0\| = n_0 \).

Note by * \( \langle A_0 \cup \hat{v}, R \rangle \models \sigma^\tau \) so we assume \( A_0 \cup \hat{v} = A \).

**Lemma 11.** \( \forall B \in [A]^{n_0} [B = A_0 \iff \forall C \in [A-B]^{n-n_0} B \cup C \in \theta^\sigma \] \)

**Proof.**

* By definition of \( A_0 \).

* Suppose \( B \neq A \). Choose \( C \in [A - (B \cup A_0)]^{n-n_0} \).

Since \( B \cap A_0 \subseteq A_0 \) has smaller cardinality (b) happens, and hence \( B \cup C = \)

\( (A_0 \cap B) \cup [(B-A_0) \cup C] \in \sim \theta^\sigma \).
Lemma 12. \( \forall B \subseteq \mathbb{V} \) infinite there is an isomorphism \( F: \langle B, \mathbb{R} \rangle \rightarrow \langle \mathbb{V}, A_0, \mathbb{R} \rangle = \mathcal{O} \) which sends \( A_0 \) into \( A_0 \).

Proof.
The fact that there is an isomorphism follows from * that it sends \( A_0 \) into \( A_0 \) is immediate from Lemma 11.

Recall
\[
\theta(\bar{x}) = \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_{f \in \mathbb{V}!} \bigwedge_{m < \omega} \psi_m(f(\bar{x})).
\]
This is equivalent to
\[
\bigwedge_{m < \omega} (\bigwedge_{x_i \neq x_j} \bigwedge_{f \in \mathbb{V}!} \bigwedge_{k < \omega} \psi_k(f(\bar{x}))) = \bigwedge_{m < \omega} \mathcal{C}_m(\mathbb{R}).
\]
Define \( \forall k < \omega \)
\[
\tau_k(x_0, \ldots, x_\omega) \equiv \bigwedge_{i \neq j} x_i \neq x_j \land \bigwedge_{m < \omega} (\bigwedge_{x_i \neq x_j < n} \bigwedge_{k < \omega} \psi_k(f(\bar{x})))
\]
\[
\tau(\bar{x}) \equiv \bigwedge_{k < \omega} \tau_k(\bar{x}).
\]
Thus each \( \tau_k \) \( \Pi^0_1 \) formula and
**\( \forall \mathcal{A} \)
\( (||\mathcal{A}|| = \mathcal{A}_0(\mathcal{A} = \mathcal{O} \leftrightarrow \exists x \in [B]^{\omega} \mathcal{A} \models \tau(x))) \)
\( \exists x \in [A]^{\omega}(\mathcal{O} \models \tau(\bar{x}) \leftrightarrow \bar{x} = A_0) \).

Lemma 13. \( \exists N < \omega \) \( (N \geq 3n_0) \) \( \forall H \in [A]^N \) there is at most one \( B \in [H]^{\omega} \) such that \( \langle H, \mathbb{R} \rangle = \tau_N(B) \)

Proof.
Let \( T \) be the following theory:

\[(a) \exists x_0, \ldots, x_k \bigwedge_{i \neq j} x_i \neq x_j \text{ for } k < \omega;\]
(b) \( \tau_k(\{b_0, \ldots, b_{n_0-1}\}) \) for \( k < \omega \);
(c) \( \tau_k(\{c_0, \ldots, c_{n_0-1}\}) \) for \( k < \omega \); and
(d) \( \{b_0, \ldots, b_{n_0-1}\} \neq \{c_0, \ldots, c_{n_0-1}\} \).

This theory must be inconsistent thus
\[ \exists N < \omega \text{ as required.} \]

Define \( \bar{a}, \bar{b} \in A^m \), \( \bar{a} \sim \bar{b} \) iff \( \forall i,j (a_i = a_j \leftrightarrow b_i = b_j) \)
\[ \Delta = \{ \phi(\bar{x}) : \phi \text{ is quantifier free formula with parameters from } A_0 \}. \]

**Lemma 14.** \( \hat{V} \) is a \( \Delta \)-indisc set over \( A_0 \) in \( \sigma V \) (that is \( \forall \theta \in \Delta \forall \bar{b}, \bar{a} \in \hat{V}^m \)
\[ \overline{a} \sim \bar{b} \rightarrow (\theta(\bar{a}) \leftrightarrow \theta(\bar{b})) \]

**Proof.**
Consider \( T_h(\langle \sigma, a \rangle_{a \in A_0}) = T \) for any linear order \( < X, \langle \rangle, ||X|| = \kappa_0 \). There exists \( \mathcal{R} \models T, X \subseteq |\mathcal{R}| \)
\[ ||\mathcal{R}|| = \kappa_0, \langle X, \langle \rangle \rangle \text{-indiscernible over } \mathcal{R}. \]

Let \( \mathcal{R}' = \langle \mathcal{R}'_0, b \rangle_{a \in A_0} \), by \( * \), \( \mathcal{R}'_0 = \sigma \) it is clear that
\[ \{b_a\}_{a \in A_0} = A_0. \]

Let \( \langle X_1, \langle \rangle \rangle \) have order type \( \omega + \omega \).
\[ X_1 = \{b_i : i < \omega + \omega\}. \]

**Claim:** \( \forall \theta \in \Delta \forall i_1 < i_2 < \ldots < i_m < k < \ell < j_1 < j_2 < \ldots < j_{m_2} \)
\[ \theta(b_{i_1}, \ldots b_{i_{m_1}}, b_{j_1}, \ldots b_{j_{m_2}}) \leftrightarrow \theta(\bar{b}_1, b_k, \bar{b}\_j) \]
Proof.

Suppose not and define
\[ p(x) \leftrightarrow \left[ \bigwedge_{a \in A_0} x \neq a \land \bigwedge_{i \leq m_1} x \neq b_i \land \bigwedge_{k \leq m_2} x \neq b_{\omega+k} \right] \]
\[ \psi(x,y) \leftrightarrow \left[ \theta(b_0, \ldots, b_{m_1-1}, x, y, b_{\omega}, \ldots, b_{\omega+(m_2-1)}) \land p(x) \land p(y) \land x \neq y \right]. \]

Let \( B_0 = \{ b_i : i < \omega + (m_2-1) \} \) by Lemma 12, \( \mathcal{L} = \langle B_0 \cup A_0, R \rangle \simeq \sigma \) sending \( A_0 \) into \( A_0 \).

But by indiscernibility
\[ \langle p^*, \psi^* \rangle \models \langle \sigma, \langle \sigma_0, \sigma_1 \rangle \rangle \text{ contradicting } \omega \text{-saturation of } \sigma. \]

\text{or } \langle \sigma, \langle \sigma_0, \sigma_1 \rangle \rangle \text{ proves Claim.}

Define \( P \subseteq S_m \) by
\[ \sigma \in P \iff \forall \theta \in \Delta \forall x_1 < x < \ldots < x_m \in X \]
\[ [\theta(x_1, x_2, \ldots, x_m) \leftrightarrow \theta(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)})]. \]

(\( \sigma, \tau \in P \implies \sigma \circ \tau \in P \)) and \( P \) contains all 2-cycles of the form \((i, i+1)\) by Claim. But these generate \( m! \)
so \( P = m! \).

Now since \( \langle A_0 \cup X, R \rangle \simeq \sigma \) the lemma follows. \( \square \)

Lemma 15. Let \( Q \) be any bijection of \( \hat{V} \) into itself.
Then the map \( F_Q \) defined on \( A_0 \cup \hat{V} \) by the identity on \( A_0 \) and \( Q \) on \( V \) is an automorphism of \( \langle A_0 \cup \hat{V}, R \rangle \).

Proof.
This is immediate from Lemma 14. \( \square \)
Lemma 16. (For N in claim 4) \( \forall H \in [A]^N \forall B \in [H]^n (\langle H, R \rangle \models \forall \tau_N (B) \equiv B = A_0) \)

Proof.

(\(\leftarrow\)) is obvious since \( \sigma_{\mathcal{I}} \models \forall \tau_N (A_0) \) and \( \forall \tau_N \) is \( \Pi_1^0 \).

(\(\rightarrow\)) If \( B \neq A_0 \), suppose \( \langle H, R \rangle \models \forall \tau_N (B) \). Let \( C = B - A_0 \).
Let \( D \in [H - (A_0 \cup B)]^{\sigma_{\mathcal{I}}} \). Define \( Q : \hat{V} \rightarrow \hat{V} \) so that \( Q \) exchanges \( C \) and \( D \) and is the identity everywhere else. By lemma 15, \( F_Q \) is an automorphism of \( \sigma_{\mathcal{I}} \), and since \( F_Q \) maps \( H \) into \( H \) it is an automorphism of \( H \). Hence we have \( \langle H, R \rangle \models \forall \tau_N (F_Q (B)) \equiv F_Q (B) \neq B \), contradicting Lemma 13.

To prove the theorem just note that \( \Sigma_1^0 \) sentence

\( \exists H \in [A]^N \exists B \in [H]^n (\langle H, R \rangle \models \forall \tau_N (B)) \)

together with the \( \Pi_1^0 \) sentence:

\( \forall H \in [A]^N \forall B \in [H]^n \langle H, R \rangle \models \forall \tau_N (B) \rightarrow \tau (B) \)

is a Scott sentence for \( \sigma_{\mathcal{I}} \). Theorem 8.

It also is not hard to show that \( \forall f \in \omega^\omega \langle \omega, f \rangle \in \Sigma_2^0 \)
implies \( \langle \omega, f \rangle \in \Delta_2^0 \). But the most general statement remains open.
§3. **Reduction of Vaught's Conjecture to $\Pi^0_2$ sentences in one binary relation**

Theorem 9. ∃ a map $: \sigma \rightarrow \sigma^*$ (effective) from first order sentences to $\Pi^0_2$ sentences in one binary relation such that $\forall \kappa \geq \omega \kappa(\sigma) = \kappa(\sigma^*) \ (\kappa(\psi) = \text{number of non-isomorphic models of } \psi \text{ of size } \kappa)$. Using same procedure it is easily shown that Vaught's conjecture for sentences of $L_{\omega_1 \omega}$ reduces to $\Pi^0_2$ sentences in one binary relation.

Description of map:
First replace $\sigma$ by one having only relation symbols.
Next reduce $\sigma$ to $\Pi^0_2$ as follows: for each subformula of $\sigma$ add a relation symbol and add axioms:

\[ \forall \bar{x}(R \theta(\bar{x})(x, \bar{x}) \leftrightarrow \exists y R_{\theta}(y, \bar{x})(y, \bar{x})) \]

\[ \forall \bar{x}(R_{\theta}(\bar{x})(x) \leftrightarrow \exists R_{\theta}(\bar{x})(x)) \]

\[ \forall \bar{x}(R_{\theta}(\bar{x}, \varphi(\bar{x}))(\bar{x}) \leftrightarrow R_{\theta}(\bar{x}) \bar{x} \iff R_{\varphi}(\bar{x}) \bar{x}) \]

Thus we obtain $\sigma_0 \Pi^0_2$ containing only relation symbols and $\forall \kappa \kappa(\sigma) = \kappa(\sigma_0)$.

Next: let $R_i(x_i - x_{\kappa_i})$, $i < n$ be the relation symbol of $\sigma_0$. 
Suppose \( \sigma_0 \equiv \forall \bar{x} \exists \bar{y} \mathcal{M}(R_1, R_i) \) where 

\( \mathcal{M} \) positive boolean. Let \( R, U, P_i^j, Q_i^j \) be new symbols.

- \( R \) binary relation
- \( U \) unary relation
- \( P_i^j \) \( i < n \) \( j < \kappa \) unary relations
- \( Q_i^j \)

We next construct \( \sigma_1 \) in this new language \( \sigma_1 \) is \( \Pi_2^0 \) and \( \kappa(\sigma_0) = \kappa(\sigma_1) \ \forall \kappa > \omega \).

\( \sigma_1 \) will be conjunction of (1) - (7)

1. \( R \) is symmetric \& irreflexive.
2. \( U, P's, Q's \) are all disjoint and everything is in one of them.
3. \( \forall x, y \) \( (Sx \land Sy) \rightarrow Rxy: \)
   \( S \in \{ U, P_i^j, Q_i^j : i < n, j < \kappa \} \).

Now we describe an interpretation

\[ \theta_{R_i}(\bar{x}) \equiv \exists \bar{y}(\bigwedge_i \bar{y}_i \neq y_j \land \bigwedge_j P_i^j(y_j) \land \bigwedge_i R(x_i, y_i) \]

\[ \bigwedge_i R(y_i, y_{i+1}) \]

\( \theta_{\neg R_i}(\bar{x}) \equiv \) same except \( Q's \) in place of \( P's \).
Write \( \theta_{R_i} (\overline{x}) \equiv \exists y \psi_{R_i} (\overline{x}, y) \) for short.

(4) \( \forall \overline{x} \exists y \ M(R_i/\theta_{R_i}, \overline{y}_{R_i}/\theta_{\overline{R}_i}) \).

(5) \( \forall \overline{x} (\theta_{R_i} (\overline{x}) \leftrightarrow \theta_{\overline{R}_i} (\overline{x})) \)

(6) \( \exists_{i<n} \exists_{j<k_i} \left[ \forall y \ P_{i} (y) \rightarrow \exists x \exists z (\psi_{R_i} (\overline{x}, y) \land y_j = y) \right] \)

This says everything not in \( U \) is being used to code.

(7) \( \exists_{i<n} \left( \forall \overline{x} \forall y \forall z \left[ (\psi_R (\overline{x}, y) \land \psi_R (\overline{x}, z)) \rightarrow y = z \right] \right) \) and \( \forall \overline{x} \forall y \forall z \left[ (\psi_{\overline{R}} (\overline{x}, y) \land \psi_{\overline{R}} (\overline{x}, z)) \rightarrow y = z \right] \)

This says codes are unique.

Thus \( \sigma_1 \) is \( \Pi^0_2 \) in language with one binary relation and finite number of unary predicates and \( \forall \kappa \geq \omega \ (\kappa(\sigma) = \kappa(\sigma_1)) \).

Relabel the language of \( \sigma_1 \) so that it is \( \{S, P_n : n < N\} \) S binary, \( P_n \) unary. Then let

\( \sigma_1 \equiv \forall \overline{x} \exists \overline{y} \hat{M}(S, P_n, \gamma P_n) \) where \( \hat{M} \) position boolean in \( (P_n, \gamma P_n) \).

Now we describe \( \sigma_2 \) in language \( R \) binary and \( U \) unary.
For $n < 2N$ let

$$\tau_n(x) = U(x) \land \exists x_1, \ldots, x_n [\forall i \neq j \neq x_i \land \exists \gamma U(x_i)]$$

- $\mathcal{M} \{R(x_i, x_j): i \equiv j \pm 1 \mod(n)\}$
- $\mathcal{M} \{\gamma R(x_i, x_j): i \neq j \pm 1 \mod(n)\}$
- $R(x, x_1)$.

$$\tau_n(x) \equiv \exists x < \psi_n(x, \bar{x})$$

$$\theta(x, y) \equiv U(x) \land U(y) \land R(x, y)$$

$\sigma_2$, conjunction of 1) $\rightarrow$ 5)

1. $R$ is symmetric and irreflexive.
2. $\forall x \exists y \mathcal{M}(S/\theta, P_i/\tau_i, \tau P_i/\tau_{N+i})$
3. $\forall i < N \forall x U(x) \rightarrow [\tau_i(x) \leftrightarrow \gamma \tau_{N+i}(x)]$.
4. $\forall z (\exists U(z) \land [\forall x, \exists \psi_n(x, \bar{x}) \land (1 \leq i \leq n \land i = z)]$

(says everything not in $U$ is being used to code).

5. $\forall x \exists \psi_n(x, \bar{x}) \Rightarrow [\exists (\gamma \psi_n(x, \bar{x}) \land \psi_n(x, \bar{y}) \land \bar{x}'' = \bar{y}'']$

To get $\sigma^*$ use reflexivity of $R$ to code $U$,

$U = \{x: R(x, x)\}$ and

$\gamma U = \{x: \gamma R(x, x)\}$. 

Remark: Vaught's conjecture for $\forall_n \exists x_n \mathcal{M} \forall \gamma y_n \theta$ where \( \theta \) quantifier free reduces to universal theories, since we
can reduce to
\[ \forall m \leq \omega \exists n_0 \forall \exists \theta \left( \forall n_0 m, \exists c_n \right) \quad c_{n_0} \text{ constants.} \]

William Hanf [16] shows Vaught's conjecture for any countable first order theory reduces to complete first-order theories in the language of one binary relation. Combined with above it easily reduces it to \( \Pi_2 \) axiomatizable complete theories in one binary relation.

§4. **The number of countable rigid models and the Barwise compactness theorem.**

In the author's abstract [18] two theorems were claimed. Unfortunately there was a mistake in the proof (pointed out by S. Shelah). Here is what remains:

**Theorem B (\( \text{CH} \)).** For \( \kappa = \aleph_1 \) if \( L \models \text{"} L_\kappa \text{ is } \Sigma_1\text{-compact} \) (\( L_\kappa \) is \( \Sigma_1\text{-compact} \)) then \( \bigvee \theta \in L_\omega \text{ a } L \) (\( \theta \) first order sentence) if \( \theta \) has exactly \( \aleph_1 \)-rigid models then \( \theta \) has an uncountable rigid model.

\( L \) is the constructible sets of Gödel. An admissible set \( M \) is \( \Sigma_1\text{-compact} \) if for every \( T \) a \( \Sigma_1 \) definable subset of \( L_\omega \cap M \), if every \( \Delta \in M \) included in \( T \) has a model then \( T \) has a model. \( \Sigma_1 \) means \( \Sigma_1 \) without parameters.
Lemma 3 is due to Mansfield (19).

Lemma 3. Suppose $A$ is $\Sigma_1$ over $(HC,\in)$ (the hereditarily countable sets) with constructible parameter. If $A \neq \emptyset$ then $|A| = 2^{\aleph_0}$. 

$\exists_\alpha, \exists^{\alpha}$ for $\alpha$ an ordinal 

$\sigma^\alpha_\delta(x)$ for $\overline{a} \in |\sigma|^{<\omega}$

$\text{Sr}(\sigma)$ Scott rank of $\sigma$

are defined in Barwise [20], p. 297-303.

**Definition:** $\sigma$ is $\omega$-rigid iff $\forall a, b \in |\sigma|$ 

$(<\sigma, a> \not\equiv_{\omega, \omega} <\sigma, b> \rightarrow a = b)$. Note that $\sigma$ $\omega$-rigid $\rightarrow$ $\sigma$ is rigid and vice versa if $\sigma$ is countable. They are not equivalent since AC allows us to find a dense $A \subseteq \mathbb{R}$ such that $(A, \prec)$ is rigid.

**Definition:** $T_\alpha(\sigma)$ for $\alpha$ an ordinal. 

$T_0(\sigma) = \{(\overline{a}, \overline{b}): (\overline{a}, \overline{b}) \in \bigcup_{n<\omega} A^n \times A^n \text{ and } a_i + b_i \text{ is a partial isomorphism}\}$. 

$T_{\alpha+1}(\sigma) = \{(\overline{a}, \overline{b}) \in T_\alpha(\sigma): \forall a \exists b < \overline{a}, \overline{b} \exists (\overline{a}, \overline{b}) \in T_\alpha(\sigma) 

\forall b \exists a < \overline{a}, \overline{b} \exists (\overline{a}, \overline{b}) \in T_\alpha(\sigma)\}$

$T_\alpha(\sigma) = \bigcap_{\beta<\alpha} T_\beta(\sigma)$ for $\alpha$ a limit.

**Lemma 4.** Suppose $\text{Sr}(\sigma) = \alpha$ then the following are equivalent: 

1. $\sigma$ is $\omega$-rigid. 

2. $\forall a, b \in |\sigma| ((|\sigma| = \sigma^\alpha_\delta(b)) \iff a = b)$
(3) \( \forall a, b \in |\sigma| (\langle a, b \rangle \in T_\alpha(\sigma) \iff a = b) \)

**Proof.**

(1) \( \iff (2) \) is just 6.3 of Barwise [20], p. 298 and the definition of \( \text{Sr}(\sigma) \).

(1) \( \iff (3) \) is proved by showing by induction on \( \beta \) that \( \forall a, b \langle a, b \rangle \in T_\beta(\sigma) \) iff \( \langle a, b \rangle \in T_\beta(\sigma) \) since \( \langle a, b \rangle \in T_\beta(\sigma) \) iff \( \langle a, b \rangle \in T_\beta(\sigma) \) the result follows.

The idea behind the proof of the next lemma was suggested to me by Charles Gray.

**Lemma 5.** If \( 2^{\lambda_\alpha} > \lambda_1 \) and \( \theta \) has exactly \( \lambda_\alpha \) rigid models all of which are countable then \( \exists \alpha \lambda_\alpha < \lambda_1 \), such that \( \sigma_\alpha \in L, |\sigma_\alpha| = \lambda_\alpha \), \( \lambda_\alpha \)'s are strictly increasing and less than \( \lambda_1 \) and \( \forall \alpha \sigma_\alpha \) is an \( \omega \)-rigid model of \( \theta \).

**pf**

For \( \sigma_\alpha \) \( \omega \)-rigid define \( \mu(\sigma_\alpha) \) the canonical model isomorphic to \( \sigma_\alpha \). Let \( \alpha = \text{Sr}(\sigma_\alpha) \)

\( |\mu(\sigma_\alpha)| = \{\sigma_\alpha(v_1), a \in |\sigma_\alpha| \}. \)

\( R^\omega(\sigma_\alpha)(\sigma_\alpha(v_1), \ldots, \sigma_\alpha(v_n)) \rightarrow R^\omega(a_1, \ldots, a_n). \)

Note that by Lemma 4, part 2) for \( \sigma_\alpha \) \( \omega \)-rigid \( \mu'(\sigma_\alpha) = \sigma_\alpha \) and \( \forall \sigma_\alpha' \) \( \omega \)-rigid ( \( \sigma_\alpha' = \sigma_\alpha \) iff \( \mu'(\sigma_\alpha) = \mu'(\sigma_\alpha') \)).

Define \( A = (\mu : (HC, \varepsilon) \models \forall \exists \sigma_\alpha, \sigma_\alpha' \hbar' \sigma_\alpha \omega \text{-rigid} \land \mu' = \mu'(\sigma_\alpha)) \). \( A \) is a \( \Sigma_1 \) HC set without parameters and
has the same cardinality as the number of countable rigid models of \( \theta \). Since \( 2^{\aleph_0} > \aleph_1 \), by Lemma 3 every member of \( A \) is constructible. The existence of the sequence described is immediate since \( |A| = \aleph_1 \).

We now write a theory \( T_{\Sigma_1} \) over \((L_\kappa, \in)\) without parameters.

Let \( R \) be the similarity type of \( \theta \), then the language of \( T \) is: \( \epsilon, c_a \) for \( a \in L_\kappa, R, \lambda \) (new individual constant). \( T \) will say the following:

1) "\( \neg \text{ZFC} \)"
2) for each \( a \in L_\kappa \) "\( \forall x \ x \in c_a \leftrightarrow \exists b \epsilon a \ x = c_b \)"
3) "\( \lambda \) is an ordinal" and for each \( \alpha < \kappa \) "\( \lambda > c_\alpha \)"
4) "\( (\lambda, R) \models \theta \)"
5) "\( (\lambda, R) \) is \( \infty \)-rigid".

Note that \((L_\kappa, \in)\) is essentially uncountable from the view point of \( L \), thus by \( \Sigma_1 \) compactness and Lemma 5 and Theorem 9.5, p. 359 of Barwise [20] \( T \) has a well-founded model \( M \). Since \( M \models \neg \text{ZFC} \), \( \alpha = \text{Sr}(\lambda, R) \in M \) and so is \( (T_\beta(\lambda, R); \beta \leq \alpha) \); hence we get an uncountable \( \infty \)-rigid model of \( \theta \). Theorem B

**Remarks:**

a) Using the fact that there are only countably many first order formulas it's easy to see that there are regular \( \Sigma_1 \),
compact cardinals less than $\aleph_{\omega_1}$. However $\kappa$ $\Sigma_1$-compact implies that $\kappa$ is inaccessible, limit of inaccessibles, etc., see Barwise [20].

Questions:

(a) If $V = L$ does there exist $T$ complete first order such that $\{\alpha \in OR : \prec L_\alpha, \varepsilon \succeq T \}$ is an unbounded subset of $\omega_1$?

(b) Does $L \models L \Sigma_1$-compact $\kappa_0 > \lambda_1$ imply that every $\Pi_1$ sentence with exactly $\aleph_1$ countable models has an uncountable model?
REFERENCES


