MATH/COMP SCI/STAT 475
Introduction to Combinatorics
Section 003

Mikhail Ivanov

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- Syllabus
- We will cover Ch 2–7 closely, Ch 8, 11, 12 partly.
- Weekly hw assignments will be graded.
- 1st assignment: Read Ch 1 "What is Combinatorics?" and do hw, will not be graded.
- Today: start Ch 2: Permutations and Combinations.
- Canvas announcements.
- Zoom?
- Are you not in campus?
Definition: Given a set $S$. A *partition* of $S$ is a decomposition of $S$ into subsets such that each element of $S$ is contained in exactly one of the subsets.
Example: \( S = \{1, 2, 3, a, b\} \).

\[
A = \{1, a\}, \quad B = \{2, b\}, \quad C = \{3\}.
\]

A, B, C is a partition of S

Notation: For a set \( S \) define

\(|S| = \text{number of elements in } S\)

"cardinality of \( S \"", "size" of \( S \)

order of \( S \)

For above example

\[
|S| = 5
\]

\[
|A| + |B| + |C| = 2 + 2 + 1 = 5
\]

This illustrates:

Addition Principle: Suppose a set \( S \) is partitioned into subsets \( S_1, S_2, \ldots, S_k \). Then

\[
|S| = |S_1| + |S_2| + \ldots + |S_k|.
\]

Definition: Given sets \( A, B \). Their Cartesian product, denoted \( A \times B \), is the set of all ordered pairs \((a, b)\) with \( a \in A \) and \( b \in B \).

Example: \( A = \{1, 2, 4\}, B = \{x, y\} \). Elements of \( A \times B \) are in the table:

\[
\begin{array}{c|cc}
\times & y & \\\hline
1 & (1, x) & (1, y) \\
2 & (2, x) & (2, y) \\
4 & (4, x) & (4, y) \\
\end{array}
\]

Observe \(|A \times B| = 6 = 3 \times 2 = |A| \cdot |B|\)

In general, for any finite sets \( A, B \)

\[
|A \times B| = |A| \cdot |B|
\]
**Example:** For an integer $n \geq 1$. Find the number of sequences
\[ a_1, a_2, \ldots, a_n \quad a_i \in \{0, 1\}, \quad 1 \leq i \leq n. \]

Define $A = \{0, 1\}$. View set of sequences as the Cartesian product

\[ A \times A \times \ldots \times A \]

\# sequences =

\[ 2 \times 2 \times 2 \times \ldots \times 2 = 2^n. \]

**Example:** A box contains

- 1 red ball
- 1 white ball
- 1 blue ball
- 1 purple ball

3 balls are withdrawn, one at a time, without replacement and color noted. How many outcomes are possible? Illustrate it with "tree diagram"

\[ \# \text{ outcomes} = 4 \cdot 3 \cdot 2 = 24 \]
Above examples illustrate the **Multiplication principle**: Consider a multi-stage experiment with \( k \) stages. Assume:

- stage 1 has \( n_1 \) outcomes
- stage 2 has \( n_2 \) outcomes (regardless of result of stage 1)
- stage 3 has \( n_3 \) outcomes (regardless of results of stages 1, 2)
- stage \( k \) has \( n_k \) outcomes (regardless of results of previous stages)

Then the total number of outcomes is

\[
\text{n}_1 \times \text{n}_2 \times \ldots \times \text{n}_k.
\]

**Example:** For an integer \( n \geq 0 \) how many subsets does the set \( \{1, 2, 3, \ldots, n\} \) have?

View the situation as an \( n \)-stage experiment in which we choose a subset.

By multiplication principle the number of subsets is

\[
2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n = \frac{18!}{16!}.
\]
**Definition:** Given sets $S \subset U$. The complement of $S$ in $U$ is

$$
\overline{S} = \{x : x \in U \text{ and } x \not\in S\} = U - S
$$

We have disjoint partition

$$
U = S \cup \overline{S}.
$$

So

$$
|U| = |S| + |\overline{S}|.
$$

In other words:

**Subtraction principle:** For $|U| < \infty$ and $S \subset U$,

$$
|\overline{S}| = |U| - |S|.
$$

**Example:** A product code consists of a string of 6 symbols (letters or numbers). How many such codes contain repeated symbols?

$$
U = \text{all codes} = \text{all 6-strings} = 26 \cdot 10 = 260
$$

$$
S = \text{codes w/o repeated symbols} \quad 26 + 10 = 36
$$

$$
|U| = 36^6
$$

$$
|S| = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31
$$

$$
|U| - |S|
$$
**Division principle:**

Given a finite set $U$ and partition $U$ into subsets with equal cardinality. Then

$$\# \text{ of subsets} = \frac{|U|}{\text{cardinality of each subset}}$$

$$K = \frac{|U|}{|S_i|}$$

**Example:** Given integers $3 \leq k \leq n$. We create a pearl necklace as follows.

Given $n$ distinct pearls. Necklace uses $k$ of these pearls. How many necklaces are possible? (Two necklaces are the same if they differ by rotation/reflection.)

$$|U| = n(n-1) - \ldots - (n-k+1)$$

$$\frac{|U|}{2k} = \frac{n(n-1) - \ldots - (n-k+1)}{2k}$$

$$n = k$$

$$\frac{n!}{2^n}$$

$$k = 2$$

$$k = 1$$
**Example:** \( v_p(n) = \alpha \) Valuation

\[ n = p^\alpha \]

\[ v_3(100!) = \left[ \frac{100}{3} \right] + \left[ \frac{100}{9} \right] + \]

\[ + \left[ \frac{100}{27} \right] + \left[ \frac{100}{81} \right] + \left[ \frac{100}{243} \right] + \ldots \]

\[ = 33 + 11 + 3 + 1 = 48 \]