**Generating functions**

Lecture 21

(Ch. 7.2)

Friday, October 23st

**Definition:** Given a sequence of real numbers \( \{h_n\}_{n=0}^{\infty} \), corresponding generating function is

\[
g(x) = \sum_{n=0}^{\infty} h_n x^n.
\]

**Example:** \( h_n = 1, \forall n \)

\[
g(x) = \sum_{n=0}^{\infty} 1 \cdot x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.
\]

**Example:** \( h_n = \frac{1}{n!}, \forall n \)

\[
g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \text{exp}(x) = e^x
\]

**Example:** \( h_n = \binom{m}{n}, \forall n \) and some integer \( m \geq 0 \).

\[
g(x) = \sum_{n=0}^{\infty} \binom{m}{n} x^n = \left(1+ x\right)^m
\]
Example: Fix integer $k \geq 1$. For $n \geq 0$ let $h_n = \# \text{ integral solutions to}$

$$x_1 + x_2 + \ldots + x_k = n$$
$$x_1, x_2, \ldots, x_k \geq 0$$

Find generating function for $\{h_n\}_{n=0}^{\infty}$.

Solution: 

$$h_n = \binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n = \frac{1}{(1-x)^k}$$

Solution 2

$$\frac{1}{1-x} = \left(1 + x + x^2 + \ldots \right)\left(1 + x^2 + x^4 + \ldots \right) \cdots \left(1 + x^k + x^{2k} + \ldots \right)$$

$$= \sum_{m_1=0}^{\infty} x^{m_1} \sum_{m_2=0}^{\infty} x^{m_2} \cdots \sum_{m_k=0}^{\infty} x^{m_k}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \right) x^n$$

$$= \sum_{n=0}^{\infty} \left(\# m_1 + m_2 + \ldots + m_k = n \quad m_i > 0 \right) x^n$$

$$= g(x)$$
Example: For the $n \geq 0$ define $h_n = \#$ integral solutions to

\[
x_1 + x_2 + x_3 + x_4 = n,
\]

\[
\begin{align*}
x_1, x_2, x_3, x_4 & \geq 0, \\
x_1 & \text{ even, } x_2 \text{ odd, } x_3 \leq 4, x_4 \geq 1.
\end{align*}
\]

Find generating function for $\{h_n\}_{n=0}^{\infty}$.

\[
g(x) = \sum_{n=0}^{\infty} h_n x^n
\]

\[
= \left(\sum_{\text{even}} x^n\right) \cdot \left(\sum_{0 \leq x_3 \leq 4} x^{n_3}\right) \cdot \left(\sum_{x_4 \geq 1} x^{n_4}\right)
\]

\[
= \left(\frac{1}{1-x^2}\right) \cdot \left(\frac{x}{1-x^2}\right) \cdot \left(\frac{1-x^5}{1-x}\right)
\]

\[
= \frac{x^2 (1-x^5)}{(1-x)^2 \cdot (1-x^2)^2}
\]
Example: Given \( n \geq 0 \). How many ways to pick \( n \) fruits from an unlimited supply of apples, oranges, bananas, pears, such that:

- \(#\) apples is even
- \(#\) oranges no more than 2
- \(#\) bananas is divisible by 3
- \(#\) pears no more than 1.

\[
g(x) = (1 + x^2 + x^4 + x^6 + \ldots) (1 + x + x^2).
\]

\[
= \frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x(1+x+x^2)}
\]

\[
= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} \frac{1}{2} (n+1) x^n = \sum_{n=0}^{\infty} (n+1) x^n
\]

\[
x_n = n+1.
\]

Solution: \( k_1 = \text{apples + pears} \)

\[
2 \left[ \sum_{n} \frac{k_1}{2} \right]
\]

\( k_2 = \text{bananas + oranges} \)

\[
n = k_1 + k_2
\]
Example: Suppose you want to create a set of weights so that any object with an integer weight from 1 to $n$ pounds can be balanced on a one-sided scale by placing a certain combination of these weights onto that scale. What is the fewest number of weights you need, and what are their weights?

$n = 3$
\[\frac{2}{1, 2}\]

\[
n \leq a_1, a_2, a_3, \ldots, a_k
\]

\[
(1 + x^{a_1})(1 + x^{a_2}) \ldots (1 + x^{a_k})
\]

\[
\leq 2^k \text{ positive coef. } 1, x^2, \ldots, x^n
\]

\[1 + n \leq 2^k \Rightarrow k > \log_2(n+1)\]

\[
(1 + x)(1 + x^2) \ldots (1 + x^n)
\]

\[
= \frac{(1 - x^2)(1 - x^3)}{(1 - x)(1 - x^2)} \ldots \frac{(1 - x^{m+1})}{(1 - x^m)}
\]

\[
= 1 - x^{m+1}
\]

\[
\frac{1}{1 - x} = 1 + x + x^2 + \ldots + x^{m-1}
\]
**Example:** Suppose you want to create a set of weights so that any object with an integer weight from 1 to \( n \) pounds can be balanced on a two-sided scale by placing a certain combination of these weights onto that scale. What is the fewest number of weights you need, and what are their weights?

\[
2n + 1 \leq 3^k \\
k \geq \log_3 (2n + 1)
\]

\[1, 3, 3^2, 3^3, \ldots, 3^m\]

\[
\frac{1}{1 + x + x^2} \left( \frac{1 - x}{1 - x^3} \right)
\]

\[
= \frac{1}{3^n - 1} \left( \frac{1 - x}{1 - x^3} \right)
\]

\[
= \frac{1 - x^{3^{m+1}}}{(1 - x)^{\frac{3^{m+1} - 1}{2}}}
\]

\[
= \frac{1 - x^{3^{m+1}}}{x^{\frac{3^{m+1} - 1}{2}}}
\]

\[
= \frac{\frac{3^n - 1}{2} - x}{x^{\frac{3^{m+1} - 1}{2}}}
\]

\[
= \frac{\frac{3^n - 1}{2}}{x^{\frac{3^{m+1} - 1}{2}}}
\]

\[
= \frac{3^n - 1}{x^{\frac{3^{m+1} - 1}{2}}}
\]