Example: Shortest cycle in general graph, multigraph, simple graph.
**Definition:** A graph $G$ is called *connected*, if for each pair of vertices $x$ and $y$, there is a walk joining $x$ and $y$ (equivalently, a trail, equivalently, a path).

$d(x, y)$ — length of shortest walk (trail, path) joining $x$ and $y$.

d is a metric, i.e.

1. $d(x, y) = 0 \iff x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

**Example:**

**Definition:** $G = (V, E)$, $V' \subseteq V$, $E' \subseteq E$. $G' = (V', E')$ is a *subgraph* of $G$.

If $E' = E \cap 2^V$ then $G'$ is an *induced subgraph*.

If $V' = V$ then $G'$ is a *spanning subgraph*.

**Example:**
**Theorem:** Let $G = (V, E)$ be a graph. Then $\exists$

$$V = V_1 \cup V_2 \cup V_3 \ldots \cup V_k, \quad G_i = G|_{V_i},$$

such that $G_i$ — connected and $\forall x \in G_i, y \in G_j, i \neq j, \not\exists$ path between $x$ and $y$.

$G_i$ — connected components.

**Proof:** $R$ is a relation on $V$

$xRy \iff \exists$ path connected $x$ and $y$ or $x = y$

$R$ — equivalence relation.

1) $xRx$
2) $xRy \iff yRx$
3) $xRy, yRz \Rightarrow xRz$

**Eulerian Trails**
The old city of Königsberg in East Prussia (now Kaliningrad, Russia) was located along the banks and on two islands of the Pregel River, with the four parts of the city connected by seven bridges. On Sundays, the citizens of Königsberg would promenade about town, and the problem arose as to whether it was possible to plan a promenade so that each bridge is crossed once and only once, ending the promenade where it began.
It can be obtained from very different situations: mail carrier, snowplow ...

G. 3? \text{ closed trail that contains all edges of } G.\text{.}

1. connectivity

2. trail \( \exists \rightarrow \deg (v_i) \) - even.

Example.

1 - 2 - 4 - 8 - 7 - 5 - 2 - 3 - 4 - 5 - 6 - 1
**Theorem:** Let $G$ be a connected graph. Then $G$ has a closed Eulerian trail if and only if the degree of each vertex is even.
Example: Is it possible to trace $n$ intersecting circles with a pencil without removing the pencil from the paper.