**Theorem:**

1. A connected graph of order \( n \) has at least \( n - 1 \) edges.
2. Removing any edge from a connected graph of order \( n \) with exactly \( n - 1 \) edges leaves a disconnected graph (each edge is the bridge).

**Definition:** A tree is a connected graph that becomes disconnected upon the removal of any edge.

**Theorem:** A connected graph is a tree if and only if \( |E| = |V| - 1 \).
**Lemma:** Let $G$ be a connected graph and let $\alpha = \{x, y\}$ be an edge of $G$. Then $\alpha$ is a bridge if and only if there does not exist a cycle of $G$ containing $\alpha$.

**Proof:**

Suppose $\alpha$ is a bridge. Then there is no cycle containing $\alpha$.

Suppose not. $G$ has a cycle containing $\alpha$.

**Theorem:** Let $G$ be a connected graph of order $n$. Then $G$ is a tree if and only if $G$ does not have any cycles.

**Theorem:** A graph $G$ is a tree if and only if every pair of distinct vertices $x$ and $y$ is joined by a unique path.
**Definition:** Let $G$ be a graph. A *pendent vertex* (or *leaf*) of $G$ is a vertex whose degree is equal to 1.

**Theorem:** Let $G$ be a tree of order $n \geq 2$. Then $G$ has at least two pendent vertices.

**Proof 1:** 
\[ d_1 + d_2 + \ldots + d_n = 2|E| = 2(n-1) \]

1 leaf \( \Rightarrow \) \( d_1, d_2, \ldots, d_{n-1} \geq 2 \)
\[ d_n = 1 \]
\[ \Rightarrow \forall d_i \geq 2n - 1 \text{. Contradiction} \]

**Proof 2:** 
1. $V_1$, $V_2$, $V_5$, $V_k$ pendent
2. $V_m$ pendent
3. $n$ pendent vertices
4. A tree with 2 pendent vertices?
**Theorem:** Every connected graph has a spanning tree.

**Example:** How many trees with 5 vertices?

2

**Remark:** G

**Spanning tree:**

**Theorem:** Every connected graph has a spanning tree.

**Example:**
**Theorem:** Let $T$ be a spanning tree of a connected graph $G$. Let $\alpha = \{a, b\}$ be an edge of $G$ that is not an edge of $T$. Then there is an edge $\beta$ of $T$ such that the graph $T'$ obtained from $T$ by inserting $\alpha$ and deleting $\beta$ is also a spanning tree of $G$.

Proof.

Let $T = T \cup \{a, b\}$.

If $\alpha \in T$, then choose $\beta \in T \setminus \alpha$, $\beta \neq \alpha$.

Then $T' = T \cup \{a, b\} \setminus \{\beta\}$ is a tree.
**Theorem:** Let $T_1$ and $T_2$ be spanning trees of a connected graph $G$. Let $\beta$ be an edge of $T_1$. Then there is an edge $\alpha$ of $T_2$ such that the graph obtained from $T_1$ by inserting $\alpha$ and deleting $\beta$ is a spanning tree of $G$.
Rooted tree