Local definability of $K$-pairs in the enumeration degrees

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joint work with H. Ganchev

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Preliminaries: The enumeration degrees

Definition

- $A \leq_e B$ iff there is a c.e. set $W$, such that $A = W(B) = \{x \mid \exists u(\langle x, u \rangle \in W \land D_u \subseteq B)\}$.
- $d_e(A) = \{B \mid A \leq_e B \land B \leq_e A\}$
- $d_e(A) \leq d_e(B)$ iff $A \leq_e B$.
- $0_e = d_e(\emptyset) = \{W \mid W \text{ is c.e.}\}$.
- $d_e(A) \lor d_e(B) = d_e(A \oplus B)$.
- $d_e(A)' = d_e(A')$, where $A' = L_A \oplus \overline{L_A}$ and $L_A = \{x \mid x \in W_x(A)\}$.
- $\mathcal{D}_e = \langle D_e, \leq, \lor, \cdot, 0_e \rangle$ is an upper semi-lattice with jump operation and least element.
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Preliminaries: The local structure

The jump operation gives rise to the local structure of the enumeration degrees $\mathcal{G}_e = \mathcal{D}_e(\leq 0'_e)$.

$\Sigma^0_2$ e-degrees

$\Delta^0_2$ e-degrees

$\Pi^0_1$ e-degrees
Preliminaries: The local structure

The local structure $\mathcal{G}_e$ can be partitioned into classes with respect to the jump hierarchy:

**Definition**

A degree $a \in \mathcal{G}_e$ is low if $a' = 0'_e$.

Or in terms of its relationship to the Turing degrees.

**Proposition**

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation:

The sub-structure of the total e-degrees is defined as $\mathcal{TOT} = \iota(\mathcal{D}_T)$.

- Every low e-degree is $\Delta^0_2$.
- Every total e-degree in $\mathcal{G}_e$ is $\Delta^0_2$.
- There are properly $\Sigma^0_2$ degrees.
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**Definition**

Let $A$ and $B$ be a pair sets of natural numbers. The pair $(A, B)$ is a $\mathcal{K}$-pair (e-ideal) if there exists a c.e. set $W$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. 
**Example**

Let $V$ be a c.e set. Then $(V, A)$ is a $\mathcal{K}$-pair for any set of natural numbers $A$.

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

We will only be interested in non-trivial $\mathcal{K}$-pairs.
\( \kappa \)-pairs: A more interesting example

**Definition (Jockusch)**

A set of natural numbers \( A \) is semi-recursive if there is a computable function \( s_A \) such that for every pair of natural numbers \((x, y)\):

1. \( s_A(x, y) \in \{x, y\} \).
2. If \( x \in A \) or \( y \in A \) then \( s_A(x, y) \in A \).

**Example**

Let \( A \) be a semi-recursive set. Then \((A, \overline{A})\) is a \( \kappa \)-pair.

**Theorem (Jockusch)**

For every noncomputable set \( B \) there is a semi-recursive set \( A \equiv_T B \) such that both \( A \) and \( \overline{A} \) are not c.e.
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An order theoretic characterization of $\mathcal{K}$-pairs

**Theorem (Kalimullin)**

$(A, B)$ is a $\mathcal{K}$-pair if and only if the degrees $a = d_e(A)$ and $b = d_e(B)$ have the following property:

$$\mathcal{K}(a, b) \iff (\forall x)((a \lor x) \land (b \lor x) = x)$$

**Question**

Is the property “$a$ and $b$ form a $\mathcal{K}$-pair” first order definable in the local structure?
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Is the property "$a$ and $b$ form a $\mathcal{K}$-pair" first order definable in the local structure?
The problem

**Theorem (Kalimullin)**

*If* \((A, B)\) *is not a \(\mathcal{K}\)-pair then there is a witness* \(C\) *computable from* \(A \oplus B \oplus K\) *such that:*

\[(d_e(A) \lor d_e(C)) \land (d_e(B) \lor d_e(C)) \neq d_e(C)\]

- If \(a\) and \(b\) are \(\Delta_2^0\) then \(C\) is also \(\Delta_2^0\) and \(\mathcal{K}(a, b)\) ensures “\(a\) and \(b\) are a true \(\mathcal{K}\)-pair”.
- Every \(\mathcal{K}\)-pair in \(\mathcal{G}_e\) consists of low (hence \(\Delta_2^0\)) e-degrees.
- If \(a\) and \(b\) are properly \(\Sigma_2^0\) then \(C\) is at best \(\Delta_3^0\). So it is possible that there is a fake \(\mathcal{K}\)-pair \(a\) and \(b\) such that

\[\mathcal{G}_e \models \mathcal{K}(a, b), \text{ but } \mathcal{D}_e \models \neg \mathcal{K}(a, b)\]
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Cupping properties

Definition

A $\Sigma^0_2$ enumeration degree $a$ is called cuppable if there is an incomplete $\Sigma^0_2$ e-degree $b$, such that $a \lor b = 0_e'$. If furthermore $b$ is low, then $a$ will be called low-cuppable.

Proposition (The $\mathcal{K}$-cupping property)

Let $a$ and $b$ are $\Sigma^0_2$ degrees such that $G_e \models \mathcal{K}(a, b)$. If $c$ is a $\Sigma^0_2$ degree, such that $c \lor b = 0_e'$ then $a \leq c$.

Proof:

$$c = (a \lor c) \land (b \lor c) = (a \lor c) \land 0_e' = a \lor c$$

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Note! If $c$ is low then $a \leq c$ is low.
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Proposition (The $\kappa$-cupping property)

Let $\mathbf{a}$ and $\mathbf{b}$ are $\Sigma^0_2$ degrees such that $G_e \models \kappa(\mathbf{a}, \mathbf{b})$.
If $\mathbf{c}$ is a $\Sigma^0_2$ degree, such that $\mathbf{c} \lor \mathbf{b} = 0'_e$ then $\mathbf{a} \leq \mathbf{c}$.

Proof:

$$\mathbf{c} = (\mathbf{a} \lor \mathbf{c}) \land (\mathbf{b} \lor \mathbf{c}) = (\mathbf{a} \lor \mathbf{c}) \land 0'_e = \mathbf{a} \lor \mathbf{c}$$

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Theorem (S, Wu)

Every nonzero $\Delta^0_2$ enumeration degree $a$ is low-cuppable, i.e. there is a low $b$ such that $a \lor b = 0'_e$.

Theorem (Cooper, Sorbi, Yi)

There are non-cuppable nonzero $\Sigma^0_2$ enumeration degrees.

Question

Are all cuppable degrees also low-cuppable?
Cupping $0_e'$-splittings

**Theorem**

If $u$ and $v$ are $\Sigma_2^0$ enumeration degrees such that $u \lor v = 0_e'$ then $u$ is low-cuppable or $v$ is low-cuppable.

**Proof:**
Uses a construction very similar to the construction of a non-splitting enumeration degree.
A non-splitting theorem

**Theorem (S)**

There is a degree \( a < 0'_e \) such that no pair of incomplete \( \Sigma_2^0 \) degrees \( u \) and \( v \) above \( a \) splits \( 0'_e \).

We build a \( \Sigma_2^0 \) set \( A \) and an auxiliary \( \Pi_1^0 \) set \( E \) so that:

\[
\mathcal{N}_\Phi : \Phi(A) \neq E
\]

\[
\mathcal{P}_\Theta, u, v : \Theta(U \oplus V) = E \Rightarrow \exists \Gamma, \land(\Gamma(U \oplus A) = \overline{K} \lor \land(V \oplus A) = \overline{K})
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**Corollary**

There exists an incomplete \( \Sigma_2^0 \) e-degree \( a \), such that for every pair of \( \Sigma_2^0 \) enumeration degrees \( u \) and \( v \) with \( u \lor v = 0'_e \) either \( u \lor a = 0'_e \) or \( v \lor a = 0'_e \).
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Theorem (S)

There is a degree $a < 0'_e$ such that no pair of incomplete $\Sigma_2$ degrees $u$ and $v$ above $a$ splits $0'_e$.

We build a $\Sigma_2^0$ set $A$ and an auxiliary $\Pi_1^0$ set $E$ so that:

$\mathcal{N}_\Phi : \Phi(A) \neq E$

$\mathcal{P}_{\Theta, u, v} : \Theta(U \oplus V) = E \Rightarrow \exists \Gamma, \Lambda(\Gamma(U \oplus A) = K \lor \Lambda(V \oplus A) = K)$

Corollary

There exists an incomplete $\Sigma_2^0$ $e$-degree $a$, such that for every pair of $\Sigma_2^0$ enumeration degrees $u$ and $v$ with $u \lor v = 0'_e$ either $u \lor a = 0'_e$ or $v \lor a = 0'_e$.
Cupping $0'_e$-splittings

**Theorem**

If $u$ and $v$ are $\Sigma^0_2$ enumeration degrees such that $u \lor v = 0'_e$ then $u$ is low-cuppable or $v$ is low-cuppable.

**Proof:**

Fix $U$, $V$ such that $U \oplus V \equiv_e \overline{K}$.

We construct an auxiliary $\Pi^0_1$ set $E$ and find an e-operator $\Theta$ such that $\Theta(U \oplus V) = E$.

First we try to construct a 1-generic $\Delta^0_2$ set $A$ such that $A \oplus U \equiv_e \overline{K}$.

If this plan fails we have acquired sufficient information to construct a 1-generic $\Delta^0_2$ set $B$ such that $B \oplus V \equiv_e \overline{K}$.
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**Proof:**

Fix $U, V$ such that $U \oplus V \equiv_e K$. We construct an auxiliary $\Pi^0_1$ set $E$ and find an $e$-operator $\Theta$ such that $\Theta(U \oplus V) = E$.

First we try to construct a 1-generic $\Delta^0_2$ set $A$ such that $A \oplus U \equiv_e \overline{K}$. If this plan fails we have acquired sufficient information to construct a 1-generic $\Delta^0_2$ set $B$ such that $B \oplus V \equiv_e \overline{K}$.
Cupping $0'_{e}$-splittings

**Theorem**

If $u$ and $v$ are $\Sigma^0_2$ enumeration degrees such that $u \lor v = 0'_{e}$ then $u$ is low-cuppable or $v$ is low-cuppable.

**Proof:**

Fix $U, V$ such that $U \oplus V \equiv_e \overline{K}$.

We construct an auxiliary $\Pi^0_1$ set $E$ and find an e-operator $\Theta$ such that $\Theta(U \oplus V) = E$.

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If this plan fails we have acquired sufficient information to construct a 1-generic $\Delta^0_2$ set $B$ such that $B \oplus V \equiv_e \overline{K}$. 
Theorem

If \( u \) and \( v \) are \( \Sigma^0_2 \) enumeration degrees such that \( u \lor v = 0'_e \) then \( u \) is low-cuppable or \( v \) is low-cuppable.

Proof:
Fix \( U, V \) such that \( U \oplus V \equiv_e \overline{K} \).

We construct an auxiliary \( \Pi^0_1 \) set \( E \) and find an e-operator \( \Theta \) such that \( \Theta(U \oplus V) = E \).

First we try to construct a 1-generic \( \Delta^0_2 \) set \( A \) such that \( A \oplus U \equiv_e \overline{K} \).

If this plan fails we have acquired sufficient information to construct a 1-generic \( \Delta^0_2 \) set \( B \) such that \( B \oplus V \equiv_e \overline{K} \).
Corollary

If \( a, b \) are nonzero \( \Sigma_2^0 \) degrees such that \( G_e \models \kappa(a, b) \) and \( a \lor b = 0'_e \) then \((a, b)\) is a true \( \kappa \)-pair.

Proof:
By the previous theorem \( a \) is low-cuppable or \( b \) is low-cuppable.

\[ b \text{ is low-cuppable } \Rightarrow a \text{ is low } \Rightarrow a \text{ is } \Delta^0_2 \Rightarrow \]
\[ a \text{ is low cuppable } \Rightarrow b \text{ is low } \Rightarrow b \text{ is } \Delta^0_2 \Rightarrow b \text{ is low cuppable.} \]

In either case both \( a \) and \( b \) are \( \Delta^0_2 \) and hence \( \kappa(a, b) \) ensures that they form a true \( \kappa \)-pair.
Defining true $\kappa$-pairs: Step 1

**Corollary**

*If $a, b$ are nonzero $\Sigma^0_2$ degrees such that $G_e \models \kappa(a, b)$ and $a \lor b = 0'^e$ then $(a, b)$ is a true $\kappa$-pair.*

**Proof:**

By the previous theorem $a$ is low-cuppable or $b$ is low-cuppable.

If $b$ is low-cuppable

$\Rightarrow \ a$ is low $\Rightarrow \ a$ is $\Delta^0_2$ $\Rightarrow$

$a$ is low cuppable $\Rightarrow \ b$ is low $\Rightarrow \ b$ is $\Delta^0_2$ $\Rightarrow \ b$ is low cuppable.

In either case both $a$ and $b$ are $\Delta^0_2$ and hence $\kappa(a, b)$ ensures that they form a true $\kappa$-pair.
Defining true $\kappa$-pairs: Step 1

**Corollary**

If $a, b$ are nonzero $\Sigma^0_2$ degrees such that $G_e \models \kappa(a, b)$ and $a \lor b = 0'_e$ then $(a, b)$ is a true $\kappa$-pair.

**Proof:**

By the previous theorem $a$ is low-cuppable or $b$ is low-cuppable.

$b$ is low-cuppable $\Rightarrow$ $a$ is low $\Rightarrow$ $a$ is $\Delta^0_2$ $\Rightarrow$

$a$ is low cuppable $\Rightarrow$ $b$ is low $\Rightarrow$ $b$ is $\Delta^0_2$ $\Rightarrow$ $b$ is low cuppable.

In either case both $a$ and $b$ are $\Delta^0_2$ and hence $\kappa(a, b)$ ensures that they form a true $\kappa$-pair.
Defining true \( K \)-pairs: Step 1

**Corollary**

*If \( a, b \) are nonzero \( \Sigma^0_2 \) degrees such that \( G_e \models K(a, b) \) and \( a \lor b = 0'_e \) then \( (a, b) \) is a true \( K \)-pair.*

**Proof:**

By the previous theorem \( a \) is low-cuppable or \( b \) is low-cuppable.

*\( b \) is low-cuppable \( \Rightarrow a \) is low \( \Rightarrow a \) is \( \Delta^0_2 \) \( \Rightarrow \)

*\( a \) is low cuppable \( \Rightarrow b \) is low \( \Rightarrow b \) is \( \Delta^0_2 \) \( \Rightarrow b \) is low cuppable.*

In either case both \( a \) and \( b \) are \( \Delta^0_2 \) and hence \( K(a, b) \) ensures that they form a true \( K \)-pair.
Corollary

If \( a, b \) are nonzero \( \Sigma^0_2 \) degrees such that \( G_e \models \mathcal{K}(a, b) \) and \( a \lor b = 0'_e \) then \((a, b)\) is a true \( \mathcal{K} \)-pair.

Proof:

By the previous theorem \( a \) is low-cuppable or \( b \) is low-cuppable.

- \( b \) is low-cuppable \( \Rightarrow \) \( a \) is low \( \Rightarrow \) \( a \) is \( \Delta^0_2 \) \( \Rightarrow \)
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In either case both \( a \) and \( b \) are \( \Delta^0_2 \) and hence \( \mathcal{K}(a, b) \) ensures that they form a true \( \mathcal{K} \)-pair.
Corollary

If \( a, b \) are nonzero \( \Sigma^0_2 \) degrees such that \( \mathcal{G}_e \models \mathcal{K}(a, b) \) and \( a \lor b = 0' \), then \( (a, b) \) is a true \( \mathcal{K} \)-pair.

Proof:

By the previous theorem \( a \) is low-cuppable or \( b \) is low-cuppable.

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In either case both \( a \) and \( b \) are \( \Delta^0_2 \) and hence \( \mathcal{K}(a, b) \) ensures that they form a true \( \mathcal{K} \)-pair.
Defining true $\mathcal{K}$-pairs: Step 1

**Corollary**

If $a, b$ are nonzero $\Sigma_2^0$ degrees such that $G_e \models \mathcal{K}(a, b)$ and $a \lor b = 0_e'$ then $(a, b)$ is a true $\mathcal{K}$-pair.

**Proof:**

By the previous theorem $a$ is low-cuppable or $b$ is low-cuppable.

- If $b$ is low-cuppable $\implies a$ is low $\implies a$ is $\Delta_2^0$ $\implies a$ is low cuppable.
- If $a$ is low cuppable $\implies b$ is low $\implies b$ is $\Delta_2^0$ $\implies b$ is low cuppable.

In either case both $a$ and $b$ are $\Delta_2^0$ and hence $\mathcal{K}(a, b)$ ensures that they form a true $\mathcal{K}$-pair.
There is a true nontrivial $\mathcal{K}$-pair $(a, b)$, such that $a \lor b = 0'_e$, so:

Theorem

The formula

$$\mathcal{L}(a) \iff a > 0_e \land (\exists b > 0_e)(\mathcal{K}(a, b) \land (a \lor b = 0'_e))$$

defines in $\mathcal{G}_e$ a nonempty set of true halves of nontrivial $\mathcal{K}$-pairs.
Defining true $\mathcal{K}$-pairs: Step 1

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Defining true $\mathcal{K}$-pairs: Step 2

Denote by $\mathcal{L}$ the definable set of all degrees $a$, such that

$$G_e \models \mathcal{L}(a).$$

**Definition**

$x$ is downwards properly $\Sigma^0_2$ every $y \in (0_e, x]$ is properly $\Sigma^0_2$.

**Example**

If $x$ is not low cuppable then it is downwards properly $\Sigma^0_2$.

If $(a, b)$ is a fake $\mathcal{K}$-pair then i.e.:

$$G_e \models \mathcal{K}(a, b), \text{ but } D_e \models \neg \mathcal{K}(a, b)$$

then $a$ and $b$ are non-low cuppable, hence downwards properly $\Sigma^0_2$, hence incomparable with every member of $\mathcal{L}$.
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Theorem

For every nonzero $\Delta^0_2$ degree $b$ there is a nontrivial $\mathcal{K}$-pair, $(c, d)$, such that

$$b \lor c = c \lor d = 0'_e.$$ 

Hence if $(a, b)$ is a true $\mathcal{K}$-pair of $\Sigma^0_2$ e-degrees (hence low and $\Delta^0_2$) we apply this theorem to get a $\mathcal{K}$-pair $(c, d)$ such that:

- $b \lor c = 0'_e$ and hence $a \leq c$.
- $c \lor d = 0'_e$ and hence $c \in \mathcal{L}$.
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- $b \lor c = 0'_e$ and hence $a \leq c$.
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- $b \lor c = 0_e'$ and hence $a \leq c$.
- $c \lor d = 0_e'$ and hence $c \in \mathcal{L}$.
Defining true $\mathcal{K}$-pairs

- If $(a, b)$ is a fake $\mathcal{K}$-pair then $a$ and $b$ are incomparable with all members of $\mathcal{L}$.
- If $(a, b)$ is a true $\mathcal{K}$-pair then $a$ is bounded by a member of $\mathcal{L}$.

Let $\mathcal{LK}(a, b) \iff \mathcal{K}(a, b) & a > 0_e & b > 0_e & \exists c (c \geq a & \mathcal{L}(c))$

**Corollary**

A pair of $\Sigma^0_2$ enumeration degrees $a, b$ forms a nontrivial $\mathcal{K}$-pair if and only if:

$$\mathcal{G}_e \models \mathcal{LK}(a, b).$$
Defining true $\mathcal{K}$-pairs

- If $\langle a, b \rangle$ is a fake $\mathcal{K}$-pair then $a$ and $b$ are incomparable with all members of $\mathcal{L}$.
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Let $\mathcal{L}K(a, b) \iff \mathcal{K}(a, b) \land a > 0_e \land b > 0_e \land \exists c (c \geq a \land \mathcal{L}(c))$

Corollary

A pair of $\Sigma^0_2$ enumeration degrees $a, b$ forms a nontrivial $\mathcal{K}$-pair if and only if:

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Defining true $\kappa$-pairs

- If $(a, b)$ is a fake $\kappa$-pair then $a$ and $b$ are incomparable with all members of $L$.
- If $(a, b)$ is a true $\kappa$-pair then $a$ is bounded by a member of $L$.

Let $L\kappa(a, b) \iff \kappa(a, b) \land a \geq 0_e \land b \geq 0_e \land \exists c (c \geq a \land L(c))$

Corollary

A pair of $\Sigma^0_2$ enumeration degrees $a, b$ forms a nontrivial $\kappa$-pair if and only if:

$$G_e \models L\kappa(a, b).$$
Application I: The complexity of $Th(G_e)$

**Theorem**

The first order theory of $G_e$ is computably isomorphic to first order arithmetic.

Given a sentence in the language of true arithmetic $\varphi$ we want to be able to computably translate it into a sentence $\varphi_e$ in the language of the $G_e$ so that:

$$\langle \mathbb{N}, +, \ast \rangle \models \varphi \text{ iff } G_e \models \varphi_e$$

1. Represent $\langle \mathbb{N}, +, \ast \rangle$ as a partial order (PO).
2. Embed this partial order in $G_e$ and code it with a finite number of parameters.
3. Find a first order condition on the parameters, which ensures that they code a SMA.
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A special type of partial order

We can represent an SMA $\langle \mathbb{N}, +, * \rangle$ as follows:

![Diagram of SMA representation]
First tool: Coding antichains

\[ \varphi_{SW}(x, a, p, q) \iff x \leq a \text{ is a minimal solution to } x \neq (x \lor p) \land (x \lor q). \]

**Theorem (Slaman, Woodin)**

If \( \{x_i \mid i \in \mathbb{N}\} \) is an antichain, uniformly enumeration reducible to a low \( a \) then there are \( \Sigma^0_2 \) e-degrees \( p \) and \( q \), such that for arbitrary \( \Sigma^0_2 \) degree \( x \)

\[ \mathcal{G}_e \models \varphi_{SW}(x, a, p, q) \iff \exists i[x_i \in x]. \]

Goal: Embed the PO so that each level is *well presented*. 
Second tool: $\mathcal{K}$-systems

**Definition**

We shall say that a system of nonzero degrees $\{a_i \mid i \in I\}$ ($|I| \geq 2$) is a $\mathcal{K}$-system, if $\mathcal{K}(a_i, a_j)$ for each $i, j \in I$, such that $i \neq j$.

- Every $\mathcal{K}$-pair is a minimal pair, hence every $\mathcal{K}$-system is an antichain.
- The degrees that form a $\mathcal{K}$-pair with a fixed degree form an ideal. Hence if $\{a_i \mid i \in I\}$ is a $\mathcal{K}$-system and $i_1 \neq i_2 \in I$ then $\{a_{i_1} \lor a_{i_2}\} \cup \{a_i \mid i \in I, i \neq i_1, i_2\}$ is a $\mathcal{K}$-system.

**Theorem**

Every non-zero $\Delta^0_2$ e-degree bounds a well-presented $\mathcal{K}$-system.
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**Theorem**

Every non-zero $\Delta_0^2$ e-degree bounds a well-presented $\mathcal{K}$-system.
Coding an SMA below any half of a $\mathcal{K}$-pair

Fix a half of a nontrivial $\mathcal{K}$-pair $a$ and a well presented $\mathcal{K}$-system below it.

We computably divide the system $\{a_i\}_{i<\omega}$ into six infinite groups.
Coding an SMA below any half of a $\mathcal{K}$-pair

The elements of $G_1$ will represent the natural numbers. There are parameters $p_0$ and $q_0$ such that $\varphi_{SW}(x, a, p_0, q_0)$ defines them.
Coding an SMA below any half of a $\mathcal{K}$-pair

L1 is constructed from lub’s of elements from G1 and G2. There are parameters $p_1$ and $q_1$ such that $\varphi_{SW}(x, a, p_1, q_1)$ defines them.
Coding an SMA below any half of a $\mathcal{K}$-pair

L2 is constructed from lub’s of elements from L1 and G3. There are parameters $p_2$ and $q_2$ such that $\varphi_{SW}(x, a, p_2, q_2)$ defines them.
Coding an SMA below any half of a $\mathcal{K}$-pair

L3 is constructed from lub’s of elements from L2 and G4. There are parameters $p_3$ and $q_3$ such that $\varphi_{SW}(x, a, p_3, q_3)$ defines them.
Coding an SMA below any half of a $\mathcal{K}$-pair

L4 is constructed from lub’s of elements from L3 and G5. There are parameters $p_4$ and $q_4$ such that $\varphi_{SW}(x, a, p_4, q_4)$ defines them.
Finally the maximal elements are constructed from lub’s of elements from L1, L2, L3, L4 and G6. $\varphi_{SW}(x, a, p_5, q_5)$ defines them.
Coding an SMA below any half of a $\mathcal{K}$-pair

parameters $a, p_0, p_1, p_2, p_3, p_4, p_5, q_0, q_1, q_2, q_3, q_4, q_5$ code a partial order, which represents a standard model of arithmetic $\mathcal{A}(a, \overline{p}, \overline{q})$. 

So the
The other direction

Given parameters \( a, \overline{p}, \overline{q} \), let we can define a first order condition \( ST_0(a, \overline{p}, \overline{q}) \) which ensures that the parameters code a structure \( \mathcal{A}(a, \overline{p}, \overline{q}) \) which is a model of arithmetic which contains a standard part.

We ask additionally that \( ST_0(a, \overline{p}, \overline{q}) \) ensures:

- \( a \) is half of a nontrivial \( \mathcal{K} \)-pair;
- The domain of \( \mathcal{A}(a, \overline{p}, \overline{q}) \) is a \( \mathcal{K} \)-system.

Let \( b \) be such that \( a \) and \( b \) are a \( \mathcal{K} \)-pair.

If the model coded below \( a \) is embedded in all models coded below \( b \), then \( \mathcal{A}(a, \overline{p}, \overline{q}) \) will be embedded into a SMA and hence will be itself a SMA.
The other direction

Given parameters $a, \bar{p}, \bar{q}$, let we can define a first order condition $ST_0(a, \bar{p}, \bar{q})$ which ensures that the parameters code a structure $\mathcal{A}(a, \bar{p}, \bar{q})$ which is is a model of arithmetic which contains a standard part.

We ask additionally that $ST_0(a, \bar{p}, \bar{q})$ ensures:

- $a$ is half of a nontrivial $\mathcal{K}$-pair;
- The domain of $\mathcal{A}(a, \bar{p}, \bar{q})$ is a $\mathcal{K}$-system.

Let $b$ be such that $a$ and $b$ are a $\mathcal{K}$-pair.

If the model coded below $a$ is embedded in all models coded below $b$, then $\mathcal{A}(a, \bar{p}, \bar{q})$ will be embedded into a SMA and hence will be itself a SMA.
The other direction

Given parameters $a, \overline{p}, \overline{q}$, let we can define a first order condition $ST_0(a, \overline{p}, \overline{q})$ which ensures that the parameters code a structure $\mathcal{A}(a, \overline{p}, \overline{q})$ which is a model of arithmetic which contains a standard part.

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- $a$ is half of a nontrivial $\mathcal{K}$-pair;
- The domain of $\mathcal{A}(a, \overline{p}, \overline{q})$ is a $\mathcal{K}$-system.

Let $b$ be such that $a$ and $b$ are a $\mathcal{K}$-pair. If the model coded below $a$ is embedded in all models coded below $b$, then $\mathcal{A}(a, \overline{p}, \overline{q})$ will be embedded into a SMA and hence will be itself a SMA.
Comparison maps

For every model $\mathcal{A}(b, p', q')$ we ask that
\[ \forall m_a \in \mathcal{A}(a, p, q) \text{ there is an } m_b \in \mathcal{A}(b, p', q') \text{ and an antichain } (y_0, y_1, \ldots, y_m) \text{ coded by parameters } c, p'' \text{ and } q'' \text{ such that:} \]
Comparison maps

If $\mathcal{A}(a, p, q)$ is an SMA then for every $\mathcal{A}(b, p', q')$ this is true.

$c = y_0 V \ldots V y_m$
Application II

If $a$ bounds a nonzero $\Delta^0_2$ degree then it bounds a nontrivial $\mathcal{K}$-pair.

If $a$ is a downwards properly $\Sigma^0_2$ degree, then it bounds no $\mathcal{K}$-pair.

Theorem

A degree $a$ is downwards properly $\Sigma^0_2$ if and only if:

$$G_e \models \forall b, c[(b \leq a \land c \leq a) \Rightarrow \neg \mathcal{K}(b, c)].$$
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**Application III**

**Definition**

\( x \) is upwards properly \( \Sigma_2^0 \) every \( y \in [x, 0'_{e}) \) is properly \( \Sigma_2^0 \).

**Example**

1. If \( a \) is a non-splitting degree then it is upwards properly \( \Sigma_2^0 \).

2. (Bereznyuk, Coles, Sorbi) For every enumeration degree \( a < 0'_{e} \) there exists an upwards properly \( \Sigma_2^0 \) degree \( c \geq a \).
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Theorem (Jockusch)

For every noncomputable set $B$ there is a semi recursive set $A \equiv_T B$ such that both $A$ and $\overline{A}$ are not c.e.

Corollary

Every nonzero total enumeration degree can be represented as the least upper bound of a nontrivial $K$-pair.

Theorem (Arslanov, Cooper, Kalimullin)

For every $\Delta^0_2$ enumeration degree $a < 0'_e$ there is a total enumeration degree $b$ such that $a \leq b < 0'_e$. 
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For every \( \Delta^0_2 \) enumeration degree \( a < 0'_e \) there is a total enumeration degree \( b \) such that \( a \leq b < 0'_e \).
So a degree $a$ is upwards properly $\Sigma^0_2$ if and only if no element above it other than $0'_e$ can be represented as the least upper bound of a nontrivial $\mathcal{K}$-pair.

**Theorem**

A degree $a$ is upwards properly $\Sigma^0_2$ if and only if:

$$G_e \models \forall c, d (L\mathcal{K}(c, d) \& a \leq c \lor d \Rightarrow c \lor d = 0'_e).$$
Dynamic characterizations

Lemma (McEvoy)

$C$ is low if and only if it has a $\Delta^0_2$ approximation $\{C^s\}_{s<\omega}$ such that for all enumeration operators $\Theta$, $\{\Theta^s(C^s)\}_{s<\omega}$ is a $\Delta^0_2$ approximation to $\Theta(C)$.

Lemma (Kalimullin)

A pair of $\Delta^0_2$ sets $A$ and $B$ form a $\mathcal{K}$-pair if and only if there are $\Delta^0_2$ approximations $\{A^s\}_{s<\omega}$ and $\{B^s\}_{s<\omega}$ to $A$ and $B$ respectively such that for all $s$:

$$A^s \subseteq A \lor B^s \subseteq B.$$
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Proposition

If \((a, b)\) is a nontrivial \(\kappa\)-pair in \(G_e\) then \(a \lor b\) is either low or total.

Proof:

Suppose that \(A\) and \(B\) are a \(\Delta^0_2\) nontrivial \(\kappa\)-pair such that \(A \oplus B\) is not low. Let \(\{A^s\}_s < \omega\) and \(\{B^s\}_s < \omega\) be their \(\kappa\)-approximations. Then \(\{A^s \oplus B^s\}_s < \omega\) is not a low approximation.

Let \(\Theta\) and \(x\) be such that at infinitely many stages \(s\):

\[ x \in \Theta^s (A^s \oplus B^s) \setminus \Theta^{s+1} (A^{s+1} \oplus B^{s+1}). \]
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If \((a, b)\) is a nontrivial \(K\)-pair in \(G_e\) then \(a \lor b\) is either low or total.

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We have an infinite computable sequence of bad stages for \(\{A^s \oplus B^s\}_{s<\omega}\):

\[s_0 < s_1 < \cdots < s_n < \cdots\]

Let \(G_A = \{n \mid A^{s_n} \subseteq A\}\) and \(G_B = \{n \mid B^{s_n} \subseteq B\}\).

- Then \(G_B = \overline{G_A}\).
- \(G_A \equiv_e A\) and \(G_B \equiv_e B\).

Hence \(A \oplus B \equiv_e G_A \oplus G_B = G_A \oplus \overline{G_A}\) and is total.
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Hence \(A \oplus B \equiv_e G_A \oplus \overline{G_A} = G_A \oplus G_B\) and is total.
Theorem (Giorgi, Sorbi, Yang)  

*Every non-low total degree bounds a downwards properly $\Sigma^0_2$ enumeration degree.*

Corollary  

*The class of the non-low total $\Sigma^0_2$ enumeration degrees is first order definable in $\mathcal{G}_e$*

**Proof:**  
A $\Sigma^0_2$ e-degree $a$ is non-low and total is and only if  
- It is the least upper bound of the members in a nontrivial $\mathcal{K}$-pair;  
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Thank you!