The automorphism group of the enumeration degrees

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Enumeration reducibility

Definition

$A \leq_e B$ if there is a c.e. set $W$, such that

$$A = W(B) = \{ x \mid \exists D ( \langle x, D \rangle \in W \land D \subseteq B ) \} .$$
Enumeration reducibility

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\]

- \( d_e(A) = \{ B \mid A \leq_e B \& B \leq_e A \} \).
- \( d_e(A) \leq d_e(B) \) if \( A \leq_e B \).
- \( 0_e = d_e(\emptyset) \) consists of all c.e. sets.
- \( d_e(A \oplus B) = d_e(A) \lor d_e(B) \).
- \( d_e(A)' = d_e(L_A \oplus \overline{L_A}) \), where \( L_A = \{ e \mid e \in W_e(A) \} \).
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\( D_e = \langle D_e, \leq, \lor, '0 \rangle \) is an upper semi-lattice with least element and jump operation.
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

**Proposition**

$A \leq_T B \iff A \oplus \overline{A}$ is c.e. in $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$. 

The embedding $\iota: \mathcal{D}_T \rightarrow \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus A)$, preserves the order, the least upper bound and the jump operation.

The substructure of the total e-degrees is defined as $\text{TOT} = \iota(\mathcal{D}_T)$.

Theorem (Selman)

$A \leq_e B$ if and only if every total enumeration degree above $B$ is also above $A$.

$\text{TOT}$ is an automorphism base for $\mathcal{D}_e$. 

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\[(\mathcal{D}_T, \leq_T, \lor', 0_T) \cong (\mathcal{TOT}, \leq_e, \lor', 0_e) \subseteq (\mathcal{D}_e, \leq_e, \lor', 0_e)\]
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**Theorem (Selman)**

$A \leq_e B$ if and only if every total enumeration degree above $B$ is also above $A$. $\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$. 
Defining the Turing jump operator

Theorem (Shore, Slaman)

The Turing jump operator is first order definable in $\mathcal{D}_T$. 

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1. The double jump is first order definable in $\mathcal{D}_T$: Slaman and Woodin’s analysis of the automorphisms of the Turing degrees and “involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic”.

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Theorem (Shore, Slaman)

The Turing jump operator is first order definable in $\mathcal{D}_T$.

1. The double jump is first order definable in $\mathcal{D}_T$: Slaman and Woodin’s analysis of the automorphisms of the Turing degrees and “involves explicit translation of automorphism facts in definability facts via a coding of second order arithmetic”.

2. An additional structural fact: for every $a \not<_{T} 0'_{T}$ there is $g$ such that $a \lor g = g''$. 
**Definition (Kalimullin)**

A pair of sets $A, B$ are called a $\mathcal{K}$-pair if there is a c.e. set $W$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.
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- A trivial example is $\{A, U\}$ and $\{U, A\}$, where $U$ is c.e.
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- A trivial example is $\{A, U\}$ and $\{U, A\}$, where $U$ is c.e.
- If $A$ is a semi-recursive set, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair.
\( \kappa \)-pairs in the enumeration degrees

**Definition (Kalimullin)**

A pair of sets \( A, B \) are called a \( \kappa \)-pair if there is a c.e. set \( W \), such that \( A \times B \subseteq W \) and \( \overline{A} \times \overline{B} \subseteq W \).

- A trivial example is \( \{ A, U \} \) and \( \{ U, A \} \), where \( U \) is c.e.
- If \( A \) is a semi-recursive set, then \( \{ A, \overline{A} \} \) is a \( \kappa \)-pair.

**Theorem (Kalimullin)**

A pair of sets \( A, B \) are a \( \kappa \)-pair if and only if their enumeration degrees \( a \) and \( b \) satisfy:

\[
\kappa(a, b) \iff (\forall x \in D_e)( (a \lor x) \land (b \lor x) = x ).
\]
\( \mathcal{K} \)-pairs are invisible in the Turing universe

- \( \mathcal{K} \)-pairs are always quasi-minimal: the only total degree below either of them is \( \mathbf{0}_e \).

There are no \( \mathcal{K} \)-pairs in the structure of the Turing degrees.
K-pairs are invisible in the Turing universe

- K-pairs are always quasi-minimal: the only total degree below either of them is 0_e.
- A consequence of the existence of nontrivial K-pairs in D_e is that the Slaman-Shore property fails, there is a degree a ≰ e 0'_e, such that for every g, a ∨ g <_e g''.
\( \mathcal{K} \)-pairs are invisible in the Turing universe

- \( \mathcal{K} \)-pairs are always quasi-minimal: the only total degree below either of them is \( 0_e \).
- A consequence of the existence of nontrivial \( \mathcal{K} \)-pairs in \( D_e \) is that the Slaman-Shore property fails, there is a degree \( a \not\leq_e 0'_e \), such that for every \( g, a \lor g <_e g'' \).
- There are no \( \mathcal{K} \)-pairs in the structure of the Turing degrees.
Theorem (Kalimullin)

$0'_e$ is the largest degree which can be represented as the least upper bound of a triple $a, b, c$, such that $K(a, b)$, $K(b, c)$ and $K(c, a)$. 
Theorem (Kalimullin)

$0'_e$ is the largest degree which can be represented as the least upper bound of a triple $a, b, c$, such that $\mathcal{K}(a, b), \mathcal{K}(b, c)$ and $\mathcal{K}(c, a)$.

Corollary (Kalimullin)

The enumeration jump is first order definable in $D_e$. 
\( \mathcal{K} \)-pairs and the definability of the enumeration jump

**Theorem (Kalimullin)**

\( \mathbf{0}'_e \) is the largest degree which can be represented as the least upper bound of a triple \( a, b, c \), such that \( \mathcal{K}(a, b), \mathcal{K}(b, c) \) and \( \mathcal{K}(c, a) \).

**Corollary (Kalimullin)**

The enumeration jump is first order definable in \( \mathcal{D}_e \).

**Theorem (Ganchev, S)**

For every nonzero enumeration degree \( u \in \mathcal{D}_e \), \( u' \) is the largest among all least upper bounds \( a \lor b \) of nontrivial \( \mathcal{K} \)-pairs \( \{a, b\} \), such that \( a \leq_e u \).
Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $0'_{e}$ is first order definable in $D_{e}(\leq 0'_{e})$.
Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $K$-pairs below $0'e$ is first order definable in $D_e(\leq 0'e)$.

Definition

A $K$-pair $\{a, b\}$ is maximal if for every $K$-pair $\{c, d\}$ with $a \leq c$ and $b \leq d$, we have that $a = c$ and $b = d$. 
Definability in the local structure of the enumeration degrees

**Theorem (Ganchev, S)**

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**Theorem (Ganchev, S)**

In $\mathcal{D}_e(\leq 0'_e)$ a degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.
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**Theorem (Ganchev, S)**

In $\mathcal{D}_e(\leq 0'_e)$ a degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.

The class of total degrees is first order definable in $\mathcal{D}_e(\leq 0'_e)$. 
Open question

We know that:

- $\mathcal{TOT} \cap D_e(\geq 0^e)$ is first order definable.

Question

Is $\mathcal{TOT}$ first order definable in $D_e$?

Recall that the total degrees are an automorphism base for $D_e$.
Open question

We know that:

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Is $\mathcal{TOT}$ first order definable in $D_e$?

Recall that the total degrees are an automorphism base for $D_e$.

A positive answer would connect the problems of the existence of a non-trivial automorphism in both structures.
One step further in the dream world

**Theorem (Ganchev,S)**

*For every nonzero enumeration degree* $u \in D_e$,

$$u' = \max \{ a \lor b \mid \mathcal{K}(a, b) \& a \leq_e u \}.$$
One step further in the dream world

Theorem (Ganchev,S)

For every nonzero enumeration degree $u \in D_e$, 

$$u' = \max \{a \vee b \mid K(a, b) & a \leq e u\}.$$

- Suppose that a degree is total if and only if it is the least upper bound of a maximal $K$-pair.
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**Theorem (Ganchev,S)**

*For every nonzero enumeration degree* $u \in \mathcal{D}_e$,

$$u' = \max \{a \lor b \mid \mathcal{K}(a, b) \& a \leq_e u\}.$$

- Suppose that a degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.
- The relation $x$ is c.e. in $u$ would also be definable for total degrees by:

$$\exists a \exists b (x = a \lor b \& \mathcal{K}(a, b) \& a \leq_e u).$$
One step further in the dream world

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- The relation $x$ is c.e. in $u$ would also be definable for total degrees by:

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- Then for total $u$, our definition of the jump would read $u'$ is the largest total degree, which is c.e. in $u$. 
Definability via automorphism analysis in $\mathcal{D}_e$


1. Coding theorem.
2. A characterization of an automorphism in terms of a countable object.
3. A finite automorphism base.
The Coding Theorem

Theorem (Slaman, Woodin)

Every countable relation on $D_e$ can be uniformly coded by parameters.

The theory of $D_e$ is computably isomorphic to second order arithmetic.

Definition

A countable relation $R \subseteq D^n_e$ is e-presented beneath a set $A$ if there is a set $W \leq_e A$ such that $R = \{(d_e(W_{i_1}(A)), \ldots, d_e(W_{i_n}(A))) | (i_1, \ldots, i_n) \in W\}$.

Theorem (Ganchev, S)

Every countable relation on $D_e(\leq_e 0'_{e})$ which is e-presented beneath a half of a $\Delta^0_2$-pair can be uniformly coded by parameters below $0'_{e}$.

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**Definition**

A countable relation $\mathcal{R} \subseteq D_e^n$ is e-presented beneath a set $A$ if there is a set $W \leq_e A$ such that

$$\mathcal{R} = \{ (d_e(W_{i_1}(A)), \ldots, d_e(W_{i_n}(A))) \mid (i_1, \ldots, i_n) \in W \}.$$
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Every countable relation on $\mathcal{D}_e$ can be uniformly coded by parameters. The theory of $\mathcal{D}_e$ is computably isomorphic to second order arithmetic.

**Definition**

A countable relation $\mathcal{R} \subseteq \mathcal{D}_e^n$ is e-presented beneath a set $A$ if there is a set $W \leq_e A$ such that

$$\mathcal{R} = \left\{ (\mathbf{d}_e(W_{i_1}(A)), \ldots, \mathbf{d}_e(W_{i_n}(A))) \mid (i_1, \ldots, i_n) \in W \right\}.$$

**Theorem (Ganchev, S)**

Every countable relation on $\mathcal{D}_e(\leq_e 0'_e)$ which is e-presented beneath a half of a $\Delta^0_2$ $\mathcal{K}$-pair can be uniformly coded by parameters below $0'_e$. 
The Coding Theorem

Theorem (Slaman, Woodin)

Every countable relation on $\mathcal{D}_e$ can be uniformly coded by parameters. The theory of $\mathcal{D}_e$ is computably isomorphic to second order arithmetic.

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Every countable relation on $\mathcal{D}_e(\leq_e 0'_e)$ which is e-presented beneath a half of a $\Delta^0_2$ $K$-pair can be uniformly coded by parameters below $0'_e$. The theory of $\mathcal{D}_e(\leq_e 0'_e)$ is computably isomorphic to first order arithmetic.
Effectively coding and decoding

Theorem (Effective Coding Theorem)

For every $n$ there is a formula $\varphi_n$, such that for every countable relation on enumeration degrees $\mathcal{R} \subseteq \mathcal{D}_e^n$ which is e-presented beneath $R$ there are parameters $\bar{p} \leq_e d_e(\mathcal{R})''$ such that $\mathcal{R} = \{ (x_1, \ldots, x_n) \mid \mathcal{D}_e \models \varphi_n(x_1, \ldots, x_n, \bar{p}) \}$. 

Theorem (Decoding Theorem)

Let $\mathcal{R} \subseteq \mathcal{D}_e^n$ be countable and coded by parameters $\bar{p}$. Let $d_e(\mathcal{P})$ be an upper bound on these parameters. Then there is a presentation $W$ of $\mathcal{R}$, such that $W \leq_e \mathcal{P}$. 

Effectively coding and decoding

Theorem (Effective Coding Theorem)

For every $n$ there is a formula $\varphi_n$, such that for every countable relation on enumeration degrees $R \subseteq D_e^n$ which is e-presented beneath $R$ there are parameters $\bar{p} \leq_e d_e(R)''$ such that

$$R = \{ (x_1, \ldots, x_n) \mid D_e \models \varphi_n(x_1, \ldots, x_n, \bar{p}) \}.$$ 

Theorem (Decoding Theorem)

Let $R \subseteq D^n_e$ be countable and coded by parameters $\bar{p}$. Let $d_e(P)$ be an upper bound on these parameters. Then there is a presentation $W$ of $R$, such that $W \leq_e P^5$. 

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Jump ideals in $\mathcal{D}_e$

**Definition**

A set of enumeration degrees $\mathcal{I} \subseteq \mathcal{D}_e$ is a jump ideal if it is downwards closed, closed under least upper bound and closed under the jump operation.

\[
\phi_J(u, u') = \max\{a \lor b \mid K(a, b) \land a \leq u\}.
\]

**Theorem**

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. For every element $u \in \mathcal{I}$ we have the following equivalence:

\[
\mathcal{I}|_J(u, u') \iff \mathcal{D}_e|_J(u, u').
\]

If $\{a, b\}$ are not a $K$-pair then there exists $x \leq a' \lor b'$ such that $x \neq (x \lor a) \land (x \lor b)$.

If $\{a, b\}$ are a $K$-pair and $a \leq u$ then $b \leq u'$.

**Corollary**

If $\rho$ is an automorphism of a jump ideal $\mathcal{I}$ then $\rho(x') = \rho(x)'$. 

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Denote by $\varphi(u, u') : u' = \max \{ a \lor b \mid \mathcal{K}(a, b) \& a \leq u\}$.

**Theorem**

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. For every element $u \in \mathcal{I}$ we have the following equivalence: $\mathcal{I} \models \varphi_J(u, u') \iff \mathcal{D}_e \models \varphi_J(u, u')$. 

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- If $\{a, b\}$ are not a $K$-pair then there exists $x \leq a' \lor b'$ such that $x \neq (x \lor a) \land (x \lor b)$.
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Denote by $\varphi(u, u') : u' = \max \{a \lor b \mid \mathcal{K}(a, b) \& a \leq u\}$.

**Theorem**

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. For every element $u \in \mathcal{I}$ we have the following equivalence: $\mathcal{I} \models \varphi_{\mathcal{J}}(u, u') \iff \mathcal{D}_e \models \varphi_{\mathcal{J}}(u, u')$.

- If $\{a, b\}$ are not a $\mathcal{K}$-pair then there exists $x \leq a' \lor b'$ such that $x \neq (x \lor a) \land (x \lor b)$.
- If $\{a, b\}$ are a $\mathcal{K}$-pair and $a \leq u$ then $b \leq u'$.
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**Theorem**

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. For every element $u \in \mathcal{I}$ we have the following equivalence: $\mathcal{I} \models \varphi_J(u, u') \iff \mathcal{D}_e \models \varphi_J(u, u')$.

- If $\{a, b\}$ are not a $\mathcal{K}$-pair then there exists $x \leq a' \lor b'$ such that $x \neq (x \lor a) \land (x \lor b)$.
- If $\{a, b\}$ are a $\mathcal{K}$-pair and $a \leq u$ then $b \leq u'$.

**Corollary**

If $\rho$ is an automorphism of a jump ideal $\mathcal{I}$ then $\rho(x') = \rho(x)'$.
Example 1: Automorphisms act locally

Let $\langle \mathbb{N}, 0, s, +, \ast, X \rangle$ be the standard model of arithmetic with one additional predicate for membership in the set $X$. 
Example 1: Automorphisms act locally

Let $\langle \mathbb{N}, 0, s, +, *, X \rangle$ be the standard model of arithmetic with one additional predicate for membership in the set $X$.

1. Coding Theorem: The structure can be coded by parameters below $X''$. 

2. Decoding Theorem: Suppose the structure is coded by parameters below $P$ then the set $X$ is enumeration reducible to $P$.

Corollary

Let $I \subseteq J$ be jump ideals in $D_e$. Let $\rho: J \to J$ be an automorphism of $J$. Then $\rho \upharpoonright I$ is an automorphism of $I$.

Fix $x \in I$. Consider $R(X) \in \rho(x)$. Find parameters $p \leq \rho(x) = \rho(x^2)$ which code $\langle \mathbb{N}, 0, s, +, *, R(X) \rangle$. Then $\rho^{-1}(p) \leq x^2$ code the same structure. Hence $\rho(x) \leq x^7$ and hence a member of $I$. 

Mariya I. Soskova (Sofia University)
Example 1: Automorphisms act locally

Let $\langle \mathbb{N}, 0, s, +, \ast, X \rangle$ be the standard model of arithmetic with one additional predicate for membership in the set $X$.

1. Coding Theorem: The structure can be coded by parameters below $X''$.

2. Decoding Theorem: Suppose the structure is coded by parameters below $P$ then the set $X$ is enumeration reducible to $P^5$. 

Corollary

Let $I \subseteq J$ be jump ideals in $D$. Let $\rho : J \to J$ be an automorphism of $J$. Then $\rho \upharpoonright I$ is an automorphism of $I$. 

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Corollary

Let \( \mathcal{I} \subseteq \mathcal{J} \) be jump ideals in \( D_e \). Let \( \rho : \mathcal{J} \to \mathcal{J} \) be an automorphism of \( \mathcal{J} \). Then \( \rho \upharpoonright \mathcal{I} \) is an automorphism of \( \mathcal{I} \).
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Fix \( x \in \mathcal{I} \). Consider \( R(X) \in \rho(x) \). Find parameters \( p \leq \rho(x)^2 = \rho(x^2) \) which code \( \langle \mathbb{N}, 0, s, +, *, R(X) \rangle \). Then \( \rho^{-1}(p) \leq x^2 \) code the same structure. Hence \( \rho(x) \leq x^7 \) and hence a member of \( \mathcal{I} \).
Example 2: Automorphisms are locally presented

Let \( C \subseteq D_e \) be countable and e-presented beneath \( C \). Let 
\( \langle \mathbb{N}, 0, s, +, *, C, \psi \rangle \) be the standard model of arithmetic together with a counting \( \psi : \mathbb{N} \rightarrow C \), arithmetically presented beneath \( C \).
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1. Coding Theorem: The structure can be coded arithmetically in $C$. 

Corollary

Let $I \subseteq J$ be jump ideals in $\mathcal{D}_e$. Let $\rho : J \rightarrow J$ be an automorphism of $J$. If $I$ is countable and e-presented beneath $I$ and $I \in J$ then $\rho | I$ is arithmetically presented in $I$. 

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1. Coding Theorem: The structure can be coded arithmetically in $C$.

2. Decoding Theorem: Given two such structures, $\langle \mathbb{N}_1, 0_1, s_1, +_1, *_1, C_1, \psi_1 \rangle$ and $\langle \mathbb{N}_2, 0_2, s_2, +_2, *_2, C_2, \psi_2 \rangle$, both coded by parameters below $P$. Then the relation $C_1 \rightarrow C_2 = \{(x, y) \mid x \in C_1 \& y \in C_2 \& \psi_1^{-1}(x) = \psi_2^{-1}(y)\}$ is arithmetically presented relative to $P$. 
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Corollary

Let $I \subseteq J$ be jump ideals in $D_e$. Let $\rho : J \rightarrow J$ be an automorphism of $J$. If $I$ is countable and e-presented beneath $I$ and $I \in J$ then $\rho \upharpoonright I$ is arithmetically presented in $I$. 
Persistent automorphisms

Definition

Let $\mathcal{I} \subseteq \mathcal{D}_e$ be countable jump ideal. An automorphism $\rho : \mathcal{I} \rightarrow \mathcal{I}$ is called persistent if for every $x \in \mathcal{D}_e$ there is a countable jump ideal $\mathcal{J}$ and an automorphism $\rho_1 : \mathcal{J} \rightarrow \mathcal{J}$ such that $\{x\} \cup \mathcal{I} \subseteq \mathcal{J}$ and $\rho_1 \upharpoonright \mathcal{I} = \rho$. 
Persistent automorphisms

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**Theorem**

Let $\mathcal{I} \subseteq \mathcal{J}$ be countable jump ideals in $\mathcal{D}_e$. Every persistent automorphism of $\mathcal{I}$ can be extended to a persistent automorphism of $\mathcal{J}$.
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**Theorem**

Let $\mathcal{I} \subseteq \mathcal{J}$ be countable jump ideals in $\mathcal{D}_e$. Every persistent automorphism of $\mathcal{I}$ can be extended to a persistent automorphism of $\mathcal{J}$.

*Note:* Every automorphism $\pi$ of $\mathcal{D}_e$ restricted to a countable ideal $\mathcal{I}$ is a persistent automorphism of $\mathcal{I}$.
Generic persistence

Definition
Let $\mathcal{I} \subseteq \mathcal{D}_e$ be a jump ideal. An automorphism $\rho : \mathcal{I} \to \mathcal{I}$ is generically persistent if for some generic extension $V[G]$ in which $\mathcal{I}$ is countable, $\rho$ is persistent.

Theorem 1
Every automorphism $\pi : \mathcal{D}_e \to \mathcal{D}_e$ is generically persistent.

2. Let $\pi$ be an automorphism of $\mathcal{D}_e$ in some generic extension $V[G]$. Then $\pi \in L(R)$.

3. Every persistent automorphism of a countable ideal $\mathcal{I} \subseteq \mathcal{D}_e$ can be extended to an automorphism $\pi$ of $\mathcal{D}_e$. 
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Arithmetically representing images of generic degrees

Theorem (Ganchev, Soskov)

Every automorphism of $D_e$ is the identity on the cone above $\emptyset^4$. 

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Theorem (Ganchev, Soskov)

Every automorphism of $D_e$ is the identity on the cone above $\emptyset^4$.

Uses a result by Richter in the Turing degrees: If the cone above $a$ is isomorphic to the cone above $b$ in the structure of the Turing degrees with jump operation then $a^2 \leq b^3$. 
Theorem (Ganchev, Soskov)

Every automorphism of $\mathcal{D}_e$ is the identity on the cone above $\emptyset^4$.

- Uses a result by Richter in the Turing degrees: If the cone above $a$ is isomorphic to the cone above $b$ in the structure of the Turing degrees with jump operation then $a^2 \leq b^3$.

Theorem

Let $\pi$ be an automorphism of $\mathcal{D}_e$. There exists an enumeration operator $\Gamma$ such that for every 8-generic total function $g$, $\pi(d_e(g)) = d_e(\Gamma(g \oplus \emptyset^4))$. 
Corollary

Let $\pi$ be an automorphism of $D_e$. There exists an arithmetic formula $\varphi$ such that $\varphi(X, Y)$ is true if and only if $\pi(d_e(X)) = d_e(Y)$. There are therefore at most countably many automorphisms of $D_e$. 
Corollary

Let $\pi$ be an automorphism of $\mathcal{D}_e$. There exists an arithmetic formula $\varphi$ such that $\varphi(X, Y)$ is true if and only if $\pi(d_e(X)) = d_e(Y)$. There are therefore at most countably many automorphisms of $\mathcal{D}_e$.

- By Rozinas every enumeration degree $a$ is the meet of two total degrees $f_1$ and $f_2$ uniformly reducible to $a''$. 
Arithmetically representing automorphisms of $\mathcal{D}_e$

**Corollary**

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- By Rozinas every enumeration degree $a$ is the meet of two total degrees $f_1$ and $f_2$ uniformly reducible to $a''$.
- Every total enumeration degree $f$ is the meet of two 8-generic degrees uniformly reducible to $f^8$. 
Automorphism bases

**Theorem**

Let $\pi$ be an automorphism of $D_e$. There exists an enumeration operator $\Gamma$ such that for every $S$-generic total function $g$,

$$\pi(d_e(g)) = d_e(\Gamma(g \oplus \emptyset^4)).$$

**Corollary**

The structure of the enumeration degrees $D_e$ has an automorphism base consisting of:

1. A single total degree $g$.
2. A single quasiminimal degree $a$.
3. The enumeration degrees below $0^8$. 

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Assigning reals

Definition

Let $T$ be a finitely axiomatizable fragment of ZFC with $\Sigma_1$ replacement and $\Sigma_1$ comprehension;
Assigning reals

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1. A countable $\omega$-model $M$ of $T$. 

Theorem
If $(M, f, I)$ is an e-assignment of reals then $D_M = I$ and $f$ is an automorphism of $I$. 

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Definition

Let $T$ be a finitely axiomatizable fragment of ZFC with $\Sigma_1$ replacement and $\Sigma_1$ comprehension; An e-assignment of reals consists of

1. A countable $\omega$-model $\mathcal{M}$ of $T$.
2. A jump ideal $\mathcal{I}$ in $\mathcal{D}_e$.

Theorem

If $(\mathcal{M}, f, \mathcal{I})$ is an e-assignment of reals then $\mathcal{D}_e = \mathcal{I}$ and $f$ is an automorphism of $\mathcal{I}$. 
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1. A countable $\omega$-model $\mathcal{M}$ of $T$.
2. A jump ideal $\mathcal{I}$ in $\mathcal{D}_e$.
3. A bijection $f : \mathcal{D}^\mathcal{M}_e \rightarrow \mathcal{I}$, such that for all $x, y \in \mathcal{D}^\mathcal{M}_e$, if $\mathcal{M} \models x \geq y$ then $f(x) \geq f(y)$.

Theorem

If $(\mathcal{M}, f, \mathcal{I})$ is an e-assignment of reals then $\mathcal{D}^\mathcal{M}_e = \mathcal{I}$ and $f$ is an automorphism of $\mathcal{I}$.
Assigning reals

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**Theorem**

If $(\mathcal{M}, f, \mathcal{I})$ is an e-assignment of reals then $D^\mathcal{M}_e = \mathcal{I}$ and $f$ is an automorphism of $\mathcal{I}$. 
An e-assignment of reals $(\mathcal{M}, f, \mathcal{I})$ is extendable if for every $z \in D_e$ there exists an e-assignment of reals $(\mathcal{M}_1, f_1, \mathcal{I}_1)$ such that $D_e^\mathcal{M} \subseteq D_e^\mathcal{M}_1$, $\mathcal{I} \cup \{z\} \subseteq \mathcal{I}_1$ and $f \subseteq f_1$. 
Definition

An e-assignment of reals \((M, f, I)\) is extendable if for every \(z \in D_e\) there exists an e-assignment of reals \((M_1, f_1, I_1)\) such that \(D_e^M \subseteq D_e^{M_1}\), \(I \cup \{z\} \subseteq I_1\) and \(f \subseteq f_1\).

Theorem

If \((M, f, I)\) is an extendible e-assignment then there is an automorphism \(\pi : D_e \rightarrow D_e\), such that for all \(x \in D_e^M\), \(\pi(x) = f(x)\).
Example 3: Interpreting automorphisms

Let \((\mathcal{M}, f, I)\) be an extendable e-assignment of reals.
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Let \((M, f, I)\) be an extendable e-assignment of reals.

1. We can interpret this structure in \(\mathcal{D}_e\).
Example 3: Interpreting automorphisms

Let $(\mathcal{M}, f, \mathcal{I})$ be an extendable e-assignment of reals.

1. We can interpret this structure in $\mathcal{D}_e$.
2. Coding Theorem: This interpretation can be coded by finitely many parameters $\bar{p}$. 

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Example 3: Interpreting automorphisms

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Theorem

Let \(g\) be the enumeration degree of an \(8\)-generic \(g \leq_e \emptyset^8\). Then the relation \(Bi(\vec{c}, d)\), stating that “\(\vec{c}\) codes a model of arithmetic with a unary predicate for \(X\) and \(d_e(X) = d\)” is definable in \(D_e\) using parameter \(g\).
Example 3: Interpreting automorphisms

Let \((\mathcal{M}, f, \mathcal{I})\) be an extendable e-assignment of reals.

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Theorem

Let \(g\) be the enumeration degree of an 8-generic \(g \leq_e \emptyset^8\). Then the relation \(\text{Bi}(\bar{c}, \bar{d})\), stating that “\(\bar{c}\) codes a model of arithmetic with a unary predicate for \(X\) and \(d_e(X) = \bar{d}\)” is definable in \(\mathcal{D}_e\) using parameter \(g\). \(\mathcal{D}_e\) is biinterpretable with second order arithmetic using parameters.
Corollary

Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.

In particular $\text{TOT}$ is definable with one parameter.

If $R$ is invariant under automorphisms then $R$ is definable without parameters in $D_e$. In particular the hyperarithmetic jump operation is first order definable in $D_e$. 
Definability in $\mathcal{D}_e$

Corollary

Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.

1. The relation $R \subseteq \mathcal{D}_e^n$ defined by

$$R(d_e(X_1), \ldots, d_e(X_n)) \leftrightarrow R(X_1, \ldots, X_n)$$

is definable in $\mathcal{D}_e$ with one parameter.

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$\text{TOT}$ is definable with one parameter.

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Definability in $\mathcal{D}_e$

Corollary

Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.

1. The relation $\mathcal{R} \subseteq \mathcal{D}_e^n$ defined by $\mathcal{R}(\text{d}_e(X_1), \ldots, \text{d}_e(X_n)) \leftrightarrow R(X_1, \ldots, X_n)$ is definable in $\mathcal{D}_e$ with one parameter.

In particular $\text{TOT}$ is definable with one parameter.
Definability in $D_e$

Corollary

Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.

1. The relation $R \subseteq D^n_e$ defined by
   
   $R(d_e(X_1), \ldots, d_e(X_n)) \leftrightarrow R(X_1, \ldots, X_n)$

   is definable in $D_e$ with one parameter.

   In particular $\text{TOT}$ is definable with one parameter.

2. If $R$ is invariant under automorphisms then $R$ is definable without parameters in $D_e$. 
Corollary

Let $R \subseteq (2^\omega)^n$ be relation definable in second order arithmetic and invariant under enumeration reducibility.

1. The relation $R \subseteq \mathcal{D}^n_e$ defined by $R(d_e(X_1), \ldots, d_e(X_n)) \leftrightarrow R(X_1, \ldots, X_n)$ is definable in $\mathcal{D}_e$ with one parameter.
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2. If $R$ is invariant under automorphisms then $R$ is definable without parameters in $\mathcal{D}_e$.
   In particular the hyperarithmetic jump operation is first order definable in $\mathcal{D}_e$. 
Thank you!