Riemann Surfaces

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0 Brief Review of material from IB Complex Analysis

A smooth function $f$ on an open subset $U \in \mathbb{C}$ is called a holomorphic function or analytic if equivalently:

(i) $f$ is complex differentiable. i.e. For each $z_0 \in U$ the derivative $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists

(ii) $f$ admits a power expansion, if we let $D(a,r) = \{ z : |z - a| < r \}$ then $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$ valued for all $z \in D(a,r)$

(ii)⇒ We can write $f(z) = g(z)(z - a)\gamma$ locally with $g(z) = \sum_{n=\gamma}^{\infty} c_n (z - a)^{n-\gamma}$ analytic on $D(a,r)$ and $g(a) = c_{\gamma} \neq 0$ This gives the "principle of isolated zeroes". If $f \neq 0$ on a domain $U \subset \mathbb{C}$ the zeroes are isolated.

Each point $z$ has an open neighbourhood $\Delta \subset U$ for which either $f|\Delta \equiv 0$ or $f$ is nowhere zero on $\Delta \setminus \{z\}$. This gives the identity principle. If two analytic functions $f, g$ on a domain $U \subset \mathbb{C}$ agree on a non-discrete subset of $U$ then $f \equiv g$ on $U$.

**Definition.** Domain= open and connected subset of $\mathbb{C}$

Suppose $f$ is analytic on a punctured disc $D^*(a, r) = \{ z \in \mathbb{C} : 0 < |z - a| < r \}$ for some $r > 0$, i.e. $f$ has an isolated singularity at $a$.

There is a Laurent expansion of $f$ at $D^*(a, r)$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

The singularity at $a$ is:

- **Removable** if $c_n = 0$ for all $n < 0$. i.e. $f$ analytic at $a$.

- **A Pole** if $\exists N > 0 : c_{-N} \neq 0$ but $c_n = 0$ for all $n < -N$. $N$ is the order of the pole.

And we can write $f(z) = (z - a)^{-N} g(z)$ with $g$ analytic on $D(a, r), g(a) = c_{-N} \neq 0$

- **Essential** if $c_n \neq 0$ for infinitely many $n < 0$.

Removable Singularity theorem(RST): Singularity is removable $\iff f$ is bounded on some punctured neighbourhood of $a$

Casarotti-Weierstrass Theorem(CWT): If $f$ has an essential singularity at $a$, then $\forall w \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \exists$ a sequence $z_n \in D^*(a, r)$ with $z_n \to a$ and $f(z_n) \to w$

Together these imply: $f$ has a pole at $a \iff f(z) \to \infty, z \to a$

**Example.** (i) $\frac{1}{z^2 - 1}$ has a pole of order 1 at $z = 0$ while $\frac{z}{z^2 - 1}$ has a removable singularity.
(ii) \( e^{\frac{1}{z}} \) has an essential singularity at \( z = 0 \).

Recall that a function \( f \) on an open set \( U \subset \mathbb{C} \) is called meromorphic if it is analytic apart from isolated singularities and they are poles.

For example \( \frac{1}{e^{\frac{1}{z}} - 1} \) is meromorphic in \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) but not meromorphic in \( \mathbb{C} \).

Poles are \( z = \frac{1}{2\pi in}, \ n \neq 0, n \in \mathbb{Z} \)

0.1 The Complex Logarithm

We will deal with "multivalued functions" which are locally analytic but it cannot be defined globally.

e.g. The complex logarithm

For a given \( z = re^{i\theta} \neq 0 \) possible values of \( \log z \) are \( \log r + i(2\pi n + \theta) \) where \( n \in \mathbb{Z} \) (possible \( w \in \mathbb{C} \) such that \( e^w = z \))

Let \( U_1 = \mathbb{C} \setminus R_{\geq 0} \). For each \( n \in \mathbb{Z} \), we can define \( \log \) analytically on \( U \) by \( \log(re^{i\theta}) = \log r + i(2\pi n + \theta) \) where \( 0 < \theta < 2\pi \). It is analytic because it is an inverse of the holomorphic map \( \exp : \{x + iy : x \in \mathbb{R}, 2\pi n < y < 2(n+1)\pi \} \rightarrow U \) and invoke the inverse function theorem.

Alternatively, we can say \( h(z) = \int_{-1}^{z} \frac{dw}{w} + (2n + 1)i \) and prove it is holomorphic with \( h'(z) = \frac{1}{2} \)

\[
\frac{(n(z+\tau) - ln(z))}{\tau} = \frac{1}{\tau} \int_{z}^{z+\tau} \frac{dw}{w} = \int_{0}^{1} \frac{dt}{z+t\tau} = \frac{1}{z+\tau} \rightarrow \frac{1}{z}
\]

Claim: \( h(z) \) is the required inverse for \( e^z \) such that \( g(z) = e^z \)

Proof. \( g(z) = \frac{h(z)}{z} \) on \( U \) and \( g'(z) = 0 \Rightarrow g = \text{constant} \ g(-1) = 1 \Rightarrow e^{h(z)} = z \)
1 Analytic Continuation in the Plane

Definition. A function element is an ordered pair \((f, U)\) where \(U\) is a domain and \(f\) is a function defined on it.

Definition. A function element in domain \(U\) is a pair \((f, D)\) where \(D\) is a subdomain of \(U\) and \(f\) is holomorphic on \(D\).

Definition. Let \((f, E), (g, E)\) be function elements in \(U\).

(i) \((g, E)\) is a direct analytic continuation of \((f, D)\) written \((f, D) \sim (g, E)\) if \(D \cap E \neq \emptyset\) and \(f = g\) on \(D \cap E\)

(ii) \((g, E)\) is an analytic continuation of \((f, D)\) along a path \(\gamma: [0, 1] \to U\) (Written \((f, D) \approx \gamma(g, E)\)) if there are function elements \((f_i, D_i)\) in \(i=1\ldots n\) and \(\gamma([t_{j-1}, t_j]) \in D_j\) \(j=1\ldots n\) and \((f, D) = (f_0, D_0) \sim (f_1, D_1) \sim \ldots \sim (f_n, D_n) = (g, E)\)

\((g, E)\) is an analytic continuation of \((f, D)\) if \(\exists\) such a path.

Then \((f, D) \approx (g, E)\)

Remark. (i) If \((f, D) \approx \gamma(g, E)\) and \((f, D) \approx \gamma(h, E)\) for some \(\gamma \Rightarrow (g = h)\)
We will prove this later when it is less messy and when we have more machinery.

(ii) \(\sim\) is not an equivalence relation. (Even if intersection is not empty.)
However \(\approx\) is

Definition. An equivalence class \(F\) under \(\approx\) is called a complete analytic function.

Example. We consider \(\text{log}\) from last time.
\[U_1 = \mathbb{C} \setminus \mathbb{R}_{\geq 0}\]
\[U = \mathbb{C}\]
Let \(J = (\alpha, \beta) = \{\theta \in \mathbb{R}, \alpha < \theta < \beta\}\)
\[D_J = \{z = re^{i\theta} \neq 0, \theta \in J\}\]
\[f_J(z) = \log r + i\theta\]
\[L_J = (f_J, D_J)\] is a function element (last time we saw \(D_J = U : J = (0, 2\pi)\))
Consider the intervals \(A = (-\frac{\pi}{2}, \frac{\pi}{2})\)
\([B = (\frac{\pi}{6}, \frac{5\pi}{6})\]
\([C = (\frac{5\pi}{6}, \frac{11\pi}{6})\]
and consider \(D_A, D_B, D_C\). We have a curve
\[\gamma: [0, 1] \to \mathbb{C}\]
\[t \mapsto e^{2\pi it}\]
\[0 = t < \frac{1}{6} < \frac{1}{2} < \frac{5}{6} < 1 = t_4\]
\(L_A \sim L_B, L_B \sim L_C\) But \(L_A \not\sim L_C\) Points in \(D_A \cap D_C\) are of the form \(re^{i\theta}\) where \(\theta \in (-\frac{\pi}{2}, -\frac{\pi}{6})\) giving \(f_A(z) = \log z + i\theta\)
And it is also of the form \(re^{i(\theta + 2\pi)}\) with \(\theta + 2\pi \in (\frac{4\pi}{3}, \frac{14\pi}{6}) \subset C\)
So \( f_C(z) = \log r + i(\theta + 2\pi) = f_A(z) + 2\pi i \)
This analytic continuation of \( L_C \) on \( D_A \) produces \( (D_A, f_A + 2\pi i) = L_{A+2\pi} \)
so \( L_{A+2\pi} \gamma L_A \), so we can go around \( \gamma \) n times in both directions to produce all possible values of \( \log \)
This gives a \( \mathcal{F} \) for example, the \( \mathcal{F} \) determined by the equivalence class of \( (f_A, D_A) \)

1.1 The Riemann surface of \( \log \)

Consider \( U = \mathbb{C} \setminus \mathbb{R}^+ \)
\( f_n(z) = \log r + i(2\pi n + \theta) \) where \( n \in \mathbb{Z}, \theta \in (0, 2\pi) \)
This gluing construction gives the "Riemann Surface" \( R \) of \( \log \) and is now a univalued function on \( R \).
\( f_n(re^{i\theta}) = \log r + i(2\pi n + \theta) \) \( 0 < \theta < 2\pi, n \in \mathbb{Z} \)
More precisely let
\[
R = \coprod_{k \in \mathbb{Z}} (\mathbb{C}^* \times \{k\})
\]
(the disjoint union)

Topology: Open sets are unions of sets of the form
\[
\begin{align*}
D((\eta, k), r) &= \{(z, k) : |z - \eta| < r\} \\
A((\eta, b), r) &= \{(z, b) : |z - \eta| < r, \text{Im}(z) < 0\} \coprod \{(z, b - 1), |z - \eta| < r, \text{Im}(z) > 0\}
\end{align*}
\]

Check this makes \( R \) into a Hausdorff connected(also path-connected) topological space with \( \pi : R \to \mathbb{C}^* \), \((z, b) \mapsto z\) continuous.

Obviously, \( \pi|_{D((\eta, b), r)} : D((\eta, b), r) \to D(\eta, r) \)
is a homeomorphism and the same holds for
\( \pi|_{A((\eta, b), r)} : A((\eta, b), r) \to D(\eta, r) \)
We capture this in the following definition.

**Definition.** A covering map of a topological space is a continuous map \( \pi : \tilde{X} \to X \) where \( \tilde{X}, X \) Hausdorff path-connected topological spaces and \( \pi \) is a local homeomorphism.
i.e. For each \( \tilde{x} \in \tilde{X} \) then there is an open neighbourhood \( \tilde{N} \) of \( \tilde{x} \) such that
\( \pi|_{\tilde{N}} : \tilde{N} \to N \) is a homeomorphism where \( N \) is an open neighbourhood of \( x = \pi(\tilde{x}) \)
\( X \) is called the base space.
\( \tilde{X} \) is called the covering space.

Warning: Terminology, don’t confuse with the covering maps in Algebraic Topology. This notion is weaker.
The map \( \pi : R \to \mathbb{C}^* \) from the example above has a property that for each \( z \in \mathbb{C}^* \), there is a disc \( D \subset \mathbb{C}^* \) such that \( z \in D \)
\[
\pi^{-1}(D) = \prod_{i \in \mathbb{Z}} D_i \text{ where } \pi|_{D_i} : D_i \to D \text{ is a homeomorphism. These are called regular and match the ones in Algebraic Topology.}
\]

We will come back to this. Define \( f : R \to \mathbb{C} \) \( f(e^{i\theta}, k) = \log r + i(2\pi k + \theta) \) where \( 0 \leq \theta < 2\pi \)

\[
\begin{array}{ccc}
R & f & \mathbb{C} \\
\downarrow \pi & \searrow \exp & \mathbb{C}^* \\
\end{array}
\]

To be upgraded to a holomorphic diagram Additional Example: We could do something quite similar with \( z^n = \exp\left(\frac{1}{n} \log z\right) \) on \( \mathbb{C}^* \), \( n \geq 0 \) (integers) Possible branches are \( r \frac{2k\pi}{n} e^{2\pi i k} \) where \( k = 0, 1, \ldots, n-1 \) This time we only need to glue \( n \) copies of \( \mathbb{C} \setminus \mathbb{R}_{\geq 0} \) and again we obtain a diagram

\[
\begin{array}{ccc}
R & f & \mathbb{C} \\
\downarrow \pi & \searrow z^n & \mathbb{C}^* \\
\end{array}
\]

### 1.2 Power Series

Recall that a power series about \( a \in \mathbb{C} \) is uniformly and absolutely convergent on any closed disc inside its radius of convergence.

If \( r < \infty \), what can we say about analytic continuations \((f, D)\) where

\[
f = \sum_{k=0}^{\infty} a_k (z - a)^k
\]

where \( D = D(a, r) \)

Without loss of generality, we will assume that \( a = 0, r = 1 \) so we work on \( \Delta = D(0, 1) \) and \( \Pi := \partial D(0, 1) \) So we consider

\[
f = \sum_{k=0}^{\infty} a_k z^k
\]

**Definition.** A point \( z \in \Pi \) is said to be regular if \( \exists g \) analytic on an open neighbourhood \( N \) of \( Z \) such that \( f = g \) on \( N \cap \Delta \)

Defining \( \tilde{f}(f) = \begin{cases} f \text{ on } D \\ g \text{ on } N \end{cases} \)

we have a direct analytic continuation to \( \Delta \cup N \) If \( z \) is not regular, it is called singular.

Note: Regular points form open sets. Singular points form a closed set.

**Example/Warning.**

1. \( z \) regular \( \implies \sum a_k z^k \) converges.
   
   e.g. \( f(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, z \in \Delta \), Every point \( z \in \Pi \setminus \{1\} \) is regular but \( \sum z^k \) diverges for all \( z \in \Pi \)
2. $\sum a_k z^k$ converges $\not\Rightarrow$ $z$ regular
$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k(k-1)}$
Absolutely convergent if all $z \in \Delta$.
1 is a singular point.
If $f$ extends analytically to a neighbourhood of 1 then $f'' = \sum_{0}^{\infty} z^k = \frac{1}{1-z}$ extends as well. Absurd

**Theorem 1.1.** Suppose $f(z) = \sum_{0}^{\infty} a_k z^k$ has radius of convergence 1 then exists at least one singular point.

**Proof.**
\exists an open disc $D_z$ such that $f$ extends to a function $f_z$ on $\Delta \cup D_z$
If $D_z \cap D_{z_2} \neq \phi$
But $f_{z_1} = f = f_{z_2}$ on the open subset $D_{z_1} \cap D_{z_2} \cap \Delta \neq \phi$ of $D_{z_1} \cap D_{z_2}$.
By the identity principle $f_{z_1} = f_{z_2}$ on $D_{z_1} \cap D_{z_2}$. Since $\Pi$ is compact we can cover it with finitely many discs $D_{z_1}, ..., D_{z_n}$ and $f$ has an analytic extension to $\Delta \cap D_{z_1} \cap ... \cap D_{z_n}$.

However $\exists \delta > 0$: $D(0, 1 + \delta) \subset \Delta \cup D_{z_1} \cup ... \cup D_{z_n}$.
And $f$ has an analytic extension to $D(0, 1 + \delta)$. This is absurd since 1 is the radius of convergence. \(\square\)

Recall from last time that $\Delta = D(0, r)$ and $\Pi = \partial D(0, r)$
$\Pi$ is called the **natural boundary** if every point is singular.

**Example.**

$$f(z) = \sum_{0}^{\infty} z^k$$

has $\Pi$ as a natural boundary.
Consider $\omega = e^{\frac{2\pi i p}{q}}$ where $p, q \in \mathbb{Z}$. We note that the set $S = \{ \omega = e^{\frac{2\pi i p}{q}} : p, q \in \mathbb{Z} \}$ is dense in $\Pi$. Since the singular points form a closed subset, it suffices to show that every point in $S$ is singular.
Take $r = 1$.

$$f(r, \omega) = \sum_{0}^{q-1} r^{k!} \omega^{k!} + \sum_{q}^{\infty} r^{k!} \to \infty \text{ as } r \to 1$$

$\Rightarrow$ Every point is singular.
Variations. 1. $f(z) = \sum_{1}^{\infty} \frac{z^k}{k!}$ also has $\Pi$ as a natural boundary because $f'$ does.

2. We can extend the notion of natural boundary to other curves (rather than $\Pi$). For example, the imaginary axis is a natural boundary for the analytic function $f(z) = \sum_{0}^{\infty} e^{kz}$ defined in $\text{Re}(z) < 0$
2 Riemann Surfaces

**Definition.** A Riemann surface \( R \) is a connected Hausdorff topological space, together with a collection of homeomorphisms \( \phi_\alpha : U_\alpha \to D_\alpha \subseteq \mathbb{C} \) with \( U_\alpha \) open and \( \phi_\alpha(U_\alpha) = D_\alpha \) open in \( \mathbb{C} \) such that

1. \( \bigcup_\alpha U_\alpha = R \)

2. If \( U_\alpha \cap U_\beta \neq \emptyset \) then \( \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \) is holomorphic.

The pair \((U_\alpha, \phi_\alpha)\) are called charts. The collection of all charts is called an atlas. The functions \( \phi_\beta \circ \phi_\alpha^{-1} \) are called the transition functions. (They are conformal equivalence.)

**Remark.** Each point \( z \in R \) has a neighbourhood holomorphic to a disc in \( \mathbb{C} \). Hence \( R \) assumed to be connected automatically gives path-connected. (\( R \) is connected and locally path-connected as \( \phi_\alpha^{-1} \) is continuous.)

**Example.**
1. \( \mathbb{R} = \mathbb{C} \) single chart \( \phi(z) = z \)
2. \( \mathbb{R} = \mathbb{C} \) single chart \( \phi(z) = z + 1 \)
**Definition.** Two atlases \(a_1, a_2\) on a topological space \(R\) are equivalent if \(a_1 \cap a_2\) is also an atlas, i.e. for any charts \((U, \phi) \in a_1\) and \((V, \psi) \in a_2\) we have \(\psi \phi^{-1}_\alpha : \phi(U \cap V) \to \psi(U \cap V)\) holomorphic.

**Example.** (See above) Atlases (1) and (2) are clearly equivalent but not equivalent to an atlas with the single chart \(\phi(z) = \bar{z}\).

Check: In definitions above equivalence of atlases, check that it is indeed an equivalence class.

An equivalence class is called a conformal structure.

**Remark.** If \(R\) is a Riemann Surface with atlas \(\{(U, \phi)\}\) and \(S\) is a connected open subset of \(R\) then \(S\) is a Riemann Surface with atlases \(\{(U \cap S, \phi|_{U \cap S})\}\).

**Example.** Every domain of \(\mathbb{C}\) is a Riemann surface.

Note: Whenever we refer to \(\mathbb{C}\) as a Riemann Surface, we mean \(\mathbb{C}\) with the standard conformal structure determined by \((\mathbb{C}, \phi(z) = z)\).

**Definition.** Let \(R\) and \(S\) be Riemann surfaces with atlases \(\{U, \phi\}\) and \(\{(V, \psi)\}\) respectively. A continuous map \(f : R \to S\) is analytic/holomorphic if \(\forall \alpha, \beta\) such that

- \(U \cap f^{-1}(V) \neq \phi\)
- \(\psi \phi^{-1}_\alpha\) is analytic on \(\phi(U \cap f^{-1}(V))\)

\(f\) is called a conformal equivalence (analytic isomorphism/holomorphism) if \(\exists\) analytic inverse \(g : S \to R\).

**Example.** \((\mathbb{C}, \phi(z) = z) \mapsto (\mathbb{C}, \psi(z) = \bar{z})\)

**Remark.** \(f : R \to S\) holomorphic if \(\forall z \in \mathbb{R}, \exists(U, \phi)\) with \(z \in U\) and \((V, \psi)\) with \(f(z) \in V\) such that \(\psi \phi^{-1} : \phi(U \cap f^{-1}(V)) \to \psi(V)\) is holomorphic. This is because if for example, \((U', \phi')\) is another chart around \(z\), then \(\psi f(\phi')^{-1} = \psi f \phi^{-1} \phi(\phi')^{-1}\).

**Lemma 2.1.** The composition of analytic \(R \xrightarrow{f} S \xrightarrow{g} G\) is analytic.

**Proof.** Write \(h = gf\). Suppose charts are \(\{(U, \phi)\}, \{(V, \psi)\}\) \(U \cap h^{-1}(W)\) is covered by \(U \cup (f^{-1}(V) \cap h^{-1}(W))\).

For \(\beta\) s.t. intersection is non-empty, we have \(\psi \phi^{-1}_\alpha : \phi(U \cap f^{-1}(V) \cap h^{-1}(W)) \to \psi(V \cap g^{-1}(W))\)
\(\theta \delta g \psi^{-1}_\beta : \psi(V \cap g^{-1}(W)) \to \theta \delta(W)\) both analytic.
So composition is analytic.
\[\Rightarrow \theta_3 g f \phi_\alpha^{-1} : \theta_3 h \phi_\alpha \text{ is analytic on each } \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta) \cap h^{-1}(W_\delta)) \text{ and these cover } \phi_\alpha(U_\alpha \cap h^{-1}(W_\delta))\]

Tautology: If \(a_1, a_2\) are atlases for \(R\) and \(\pi : (R, a_1) \to (Ra_2)\) holomorphic \(\Rightarrow a_1, a_2\) equivalent.

If \(f : (R, a_1) \to (S, a_2)\) analytic then \(a_1 \sim a'_1\) then \(f' : (R, a_1) \to (S, a'_1)\) analytic.
\[f'(R, a'_1) \xrightarrow{\sim} (R, a_1) \xrightarrow{f} (S, a_2) \xrightarrow{\sim} (S, a'_2)\]
Can move between equivalent atlases, equivalence of atlas is an equivalence relation.

**Remark.** Sufficient to check \(\forall z \in R, \exists \text{ chart about } z \in U, \text{ chart } (U, \phi) \text{ and chart } (V, \psi) \text{ s.t. } V \ni f(z), \psi f \phi^{-1} \text{ holomorphic. } \phi(U \cap f^{-1}(N)) \ni \psi(V). \text{ Then if } (U', \phi') \text{ another chart at } z, (V', \psi') \text{ at } f(z) \text{ then } \psi' f(\phi')^{-1} = (\psi' \psi^{-1})(\psi f \phi^{-1})(\phi(\phi')^{-1}) \text{ is holomorphic.}

**Lemma 2.2.** If \(R\) is a Riemann surface and \(\pi : \tilde{R} \to R\) is a covering map of topological spaces, then there exists a unique conformal structure on \(\tilde{R}\) such that \(\pi : \tilde{R} \to R\) is analytic.

**Proof.** Given a point \(z \in \tilde{R}\), choose \(\tilde{N} \ni z \in \tilde{N}, p\tilde{u}|_{\tilde{N}} : \tilde{N} \to \pi(\tilde{N})\) is a homeomorphism.

Let \((V, \psi)\) be any chart of \(R\) with \(\pi(z) \in V\) and take chart \(\phi = \psi \pi : U \to \psi(V \cup V)\).

This chart \((U, \phi)\) form an atlas \(\tilde{a}\) of \(\tilde{R}\) transition functions are just restriction of transition functions of \(R\).

Let us check \(\pi\) analytic.
\[
\psi \pi \phi^{-1} : \psi \pi (\psi \pi)^{-1} = \psi \pi \pi^{-1} \psi^{-1} = id
\]
holomorphic on some appropriate domain.

**Uniqueness:** Suppose \(a^* = \{(W_\alpha, \theta_\alpha)\}\) is an atlas on \(\tilde{R}\) s.t.
\(p\tilde{u} : ((R), a^*) \to (R, a)\) is analytic.

We wish to show that \(a^*\) is equivalent to \(\tilde{a}\).

Given \((U, \phi) \in \tilde{a}\) as above. Assume that \(\pi|_{\tilde{N}} : \tilde{N}, a^*|_{\tilde{N}} \to (N, a|_{N})\) is holomorphic.

Given \((W_\alpha, \theta_\alpha) \in a^*\) s.t. \(W_\alpha \cap \tilde{N} \cap \pi^{-1}(V) \neq \phi\) have a diagram
\[
\begin{array}{ccc}
W_\alpha \cap U & \xrightarrow{\pi} & N \cap V \\
\downarrow{\theta_\alpha} & \phi & \downarrow{\psi} \\
\theta_\alpha(W_\alpha \cap U) & \xrightarrow{\phi} & \psi(N \cap V)
\end{array}
\]

By(*) \(\psi \pi \theta_\alpha^{-1}\) is holomorphic.

This means \(\phi \theta_\alpha^{-1}\) is holomorphic. i.e. \(a^*\) is equivalent to \(\tilde{a}\). \(\square\)

11
Example. Let us revisit log. Two lectures ago we constructed,

\[
\begin{array}{ccc}
R & \xrightarrow{f} & \mathbb{C} \\
\downarrow \pi & & \downarrow \exp \\
\mathbb{C}^* & \xrightarrow{\exp} & \\
\end{array}
\]

\[R = \coprod_{k \in \mathbb{Z}} \mathbb{C}^* \times \{k\}\] By Lemma 2.2, there exists a unique conformal structure on \(R\) making \(\pi\) holomorphic.

\((\mathbb{C}^*, \psi(z) = z)\). Charts on \(R\) are making \((U, \phi)\) where \(\phi = \pi_N\).

To check that \(\tilde{f}\) is holomorphic we only need to check that \(f \circ (\pi|_N)^{-1}\) is holomorphic.

But this is just a branch of log.

In fact, \(f\) is a bijection and its inverse is also holomorphic.

Since \(\exp(f(z)) = \pi(z)\), locally \((\pi|_N)^{-1} \circ \exp\) and this is holomorphic.

Similarly, with \(z^n\) we get a \(n\)-folded cover.

\[
\begin{array}{ccc}
R & \xrightarrow{f} & \mathbb{C} \\
\downarrow \pi & & \downarrow z^n \\
\mathbb{C}^* & \xrightarrow{z^n} & \\
\end{array}
\]

and using Lemma 2.2, we can upgrade the diagram to a diagram of holomorphic maps between Riemann Surfaces.

Important Examples: Riemann sphere, \(\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}\)

Topology: Open subsets are either open subset of \(\mathbb{C}\) or of the form \(\{\infty\} \cap (\mathbb{C} \setminus \text{compact set}\)\)

With this topology, \(\mathbb{C}_\infty\) is homeomorphic to \(S^2 = \{x \in \mathbb{R}^3 : ||x|| = 1\}\)

Homeomorphism is given by the stereographic projection \(\pi : S^2 \to \mathbb{C}_\infty\). In particular, \(\mathbb{C}_\infty\) is compact, connected and Hausdorff.

Atlas: \((\mathbb{C}, \phi_1), \phi_1(z) = z, (\mathbb{C} \setminus \{0\}, \phi_2), \phi_2(z) = \frac{1}{z}\)

Transition Function:

\[
\mathbb{C}^* \to \mathbb{C}^* \\
z \mapsto \frac{1}{z}
\]

Alternatively, we can define charts in \(S^2\) directly.

\(U_1 = S^2 \setminus \{(0,0,1)\}, \pi_1\) projection from \((0,0,1)\).

\(U_2 = S^2 \setminus \{(0,0,-1)\}, \pi_2\) projection from \((0,0,-1)\).

Recall \(\pi_1(x) \pi_2(x) = 1\)

\(\pi_1 = \frac{x + iy}{1 - z}, \pi_2 = \frac{x - iy}{1 + z}\)

Consider atlas \(\{(U_1, \pi_1), (U_2, \pi_2)\}\).

The harmonic function is again.

Complex Tori \(\mathbb{C}\) Consider a lattice
\[ \Lambda = \tau_1 \mathbb{Z} + \tau_2 \mathbb{Z} \]
\[ \tau_1, \tau_2 \neq 0, \tau_1/\tau_2 \notin \mathbb{R} \]
\[ \pi : \mathbb{C} \to \mathbb{C}/\Lambda \] We endow \( \mathbb{C}/\Lambda \) with the quotient topology.
This makes \( \pi \) continuous.
\( \mathbb{C}/\Lambda \) is a compact connected Hausdorff topological space.
For any \( a \in T \), choose \( w \in \mathbb{C} \) s.t. \( \pi(U) \ni a \) and open neighbourhood \( N \ni W \) s.t. \( \pi|_N \) is a homeomorphism.
Take as chart: \( \phi = (\pi|_N)^{-1} : U \to N \)
Two such charts \( \phi_1 : U_1 \to N_1 \)
\( \phi_2 : U_2 \to N_2 \) with \( U_1 \cap U_2 \neq \phi \)
give rise to a transition function \( \phi_2 \phi^{-1} : \phi_1(U_1 \cap U_2) \to \phi(U_1 \cap U_2) \) which is locally just a translation by some \( w \in \Lambda \) hence analytic. (Example sheet 1).

**Theorem 2.3** (Inverse Function Theorem). Given any analytic function \( g \) on an open set \( V \subset \mathbb{C} \) and \( a \in V \) such that \( g'(a) \neq 0 \), there exists an open neighbourhood \( N \) of \( a \) such that \( g|_N : N \to g(N) \) is a conformal equivalence of \( N \) onto the open neighbourhood \( g(N) \) of \( g(a) \).

**Proof.** By considering \( g(z) - g(a) \), we may assume without loss of generality that \( g \) has a simple zero at \( a \)
Take a small disc \( D = D(a, \epsilon) \) with \( \bar{D} \subset V \) such that
- \( g \) is nowhere zero in \( \bar{D} \setminus \{a\} \)
- \( g \) is nowhere zero in \( D \)

Principle of Argument \( \Rightarrow n(g \circ \gamma, 0) = 1 = \) zeroes of \( g \) in \( D \)
where \( \gamma \) a closed loop around \( a \).
Note that \( g \circ \gamma([0, 1]) \) compact hence closed.
Therefore, \( \mathbb{C} \setminus g(\gamma([0, 1])) \) is open and choose \( \Delta \) disc centred at 0 contained in the connected component containing \( g(z) \).
Hence \( n(g \circ \gamma, w) = 1 \) for all \( w \in \Delta \)
By the principle of a argument again, \( N = (g|\Delta)^{-1}(\Delta) \) is an open neighbourhood of \( a \) and \( g|_N : N \to \Delta \) is bijective.
Then, the opening mapping theorem, (IB Complex Analysis) \( \Rightarrow h = (g|_N)^{-1} \) is continuous and hence \( g|_N \) is a homeomorphism.
Pick \( w_0 \in \Delta \) and \( w \neq w_0 \) and let \( z_0 = h(w_0) \) and \( z = h(w) \) and consider
\[ \frac{h(w_0) - h(w)}{w_0 - w} = \frac{z_0 - z}{g(z_0) - g(z)} \rightarrow 1 \]
Since \( w \rightarrow w_0 \) iff \( z \rightarrow z_0 \) as \( w \rightarrow w_0 \).
(\( g|_N \) is a homeomorphism.)
2.1 Local Form of Holomorphic Function

Suppose we have \( a \in U \subset \mathbb{C} \) and \( f \not\equiv 0 \) is analytic in the domain \( U \) with \( f(a) = 0 \). Then we know we can write locally around \( a \)

\[
f(z) = (z - a)^r h(z)
\]

where \( h \) is analytic, \( h'(a) \neq 0 \) and \( r > 0 \). We can find a disc \( D \) containing \( a \) such that \( h(D) \subset \mathbb{C} \setminus \mathbb{R}_{\geq 0}e^{i\alpha} \). So on \( D \) we can choose an analytic function \( l(z) = |h(z)|^{\frac{1}{r}} = \exp(\frac{1}{r} \log(h(z))) \). Hence \( f(z) = |g(z)|^r \) where \( g(z) = (z - a)l(z) \).

By Theorem 2.3, \( g \) is locally a conformal equivalence which means if we said \( w = g(z) \) then \( f(g^{-1}(w)) = w^r \) More generally, if \( f : R \rightarrow \mathbb{C} \) locally around a point \( p_0 \in R \) we can choose a chart \( \phi : U \rightarrow \mathbb{C} \) and by replacing \( f \) by \( f(p) - f(p_0) \) we may assume \( f(p_0) = 0 \). If \( a = \phi(p_0) \) we can locally write the function \( f^{\phi^{-1}} \) as \( [g(z)]^n \) as above.

Locally around \( p_0 \) we have a chart \( \psi = g\phi \) and a diagram.

\[\psi f = \phi \]

Finally, if \( f : R \rightarrow S \) when \( R \) and \( S \) are both Riemann Surfaces take a chart \( (v, \theta) \) around \( f(p_0) \) with \( \theta(f(p_0)) = 0 \) and apply the above to \( \theta f \) so locally we have that \( f \) is of the form \( z \mapsto z^r \).

**Theorem 2.4 (Open Mapping Theorem).** Let \( f : R \rightarrow S \) be a non-constant analytic map of Riemann Surfaces. Then \( f \) is an open map.

**Proof.** Suppose \( W \) is open in \( R \), \( z \in W \). We are required to prove that \( f(W) \) contains an open neighbourhood of \( f(z) \). Choose charts

\[(U, \phi) \text{ in } R \text{ with } z \in U \]

\[(V, \psi) \text{ in } S \text{ with } f(z) \in V \]

Let \( D \) be an open disc with \( \phi(D) \subset U \cap W \cap f^{-1}(V) \).

\( \psi f \phi^{-1} \) is non-constant.

(If not by the identity principle of the Riemann Surfaces on Example Sheet 1 Q 11, \( f \) is constant on \( R \).)

By the open mapping theorem for plane domains (IB) \( \psi f \phi^{-1}(D) \) is open.

Have \( f \phi^{-1}(D) \) is open in \( S \). Also, \( f(z) \in f(\phi^{-1}(D)) \subset f(W) \) in \( S \).

Hence \( f \phi^{-1}(D) \) is the required open neighbourhood.

\[\square\]

**Corollary 2.5.** Let \( f : R \rightarrow S \) be analytic and non-constant. If \( R \) is compact then \( f(R) = S \) and \( S \) is also compact.

**Proof.** By (2.4) and \( f(R) \) is open. Since \( R \) is compact, \( f(R) \) is compact and since \( S \) is Hausdorff. \( f(R) \) is closed. Since \( S \) is connected and \( f(R) \neq \phi \) we have \( f(R) = S \).

\[\square\]

**Corollary 2.6.** Any analytic function \( f : R \rightarrow S \) with \( R \) compact must be constant.

**Proof.** Since \( \mathbb{C} \) is not compact, thus it follows from (2.5)

\[\square\]

**Remark/Application:** Suppose \( f : R \setminus \{a\} \rightarrow \mathbb{C} \) is analytic and \( a \in \mathbb{R} \).

Taking local charts around \( a \), we see \( f \) has a removable singularity \( \Leftrightarrow f \) is
bounded in a neighbourhood of \( a \).
For instance, \( f : \mathbb{C} \to \mathbb{C} \) holomorphic and bounded. Then \( f \) extends to a holomorphic map \( f : \mathbb{C}_\infty \to \mathbb{C} \) and hence is constant. (i.e. Liouville’s theorem)
2.2 Harmonic Functions

**Definition.** Let $D$ be a plane domain and $u: D \to \mathbb{R}$ is harmonic in $u \in C^2(D)$
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
Laplace’s equation.

**Remark.** If $f = u + iv$ is analytic in $D$ then $u, v$ harmonic. (just use Cauchy-Riemann equations.)
If $D$ is a disc and $u$ is harmonic then there is a holomorphic function, $f: D \to \mathbb{C}$ s.t. $u = \text{Re}(f)$. (Example sheet 1, Q12)

**Definition.** A harmonic function $h$ or a Riemann surface $R$ is a continuous function $f: R \to \mathbb{R}$ s.t. for any chart $\phi: U \to \phi(U) \subset \mathbb{C}$ on $R$ the function $h\phi^{-1}$ is harmonic on $\phi(U)$.

The remark above implies that this definition is independent for the choice of charts.
Moreover, a continuous function $h: R \to \mathbb{R}$ is harmonic iff each $z \in R$ has a neighbourhood $N$ s.t. $h|_N = \text{Re}(g)$ for some analytic function $g$ on $N$.

**Proposition 2.7.** Non-constant harmonic functions $h: R \to \mathbb{R}$ on a Riemann surface are open maps. Hence if $R$ is compact all harmonic functions are constant.

**Proof.** Suppose that $U \subset \mathbb{R}$ is open.
For each $z \in U$, there exists open neighbourhood $N \subset U$ s.t. $z \in N$ and $h = \text{Re}(g)$ where $g: N \to \mathbb{C}$ holomorphic.
$g$ is non-constant.
If $g$ is constant $\Rightarrow h$ is constant on open set.
Example sheet 1 Q 13 $\Rightarrow h$ globally constant.
Open mapping theorem for $g \Rightarrow \exists$ neighbourhood $g(z) = a + ib$ of the form.
$(a - \epsilon, a + \epsilon) \times (b - \epsilon, b + \epsilon) \subset g(N)$.
Thus $(a - \epsilon, a + \epsilon) \subset h(N) \subset h(U) \Rightarrow h(U)$ open.
If $R$ is compact and $h$ is non-constant then $h(R)$ is both open and closed.
$\Rightarrow h(R) = \mathbb{R}$ is compact. $\square$
2.3 Meromorphic functions

Definition. A meromorphic function $f$ on a Riemann Surface $R$ is an analytic map $f : R \to \mathbb{C}_\infty$

Lemma 2.8. If $U \subset \mathbb{C}$ is a plane domain, then this definition coincides with the usual one of isolated poles.

Proof. Suppose $f : U \to \mathbb{C}_\infty$ analytic.
If $a \in U$ has $f(a) \in \mathbb{C}$ then on each neighbourhood $N$ of $a$, $f$ is an analytic function on $N$. If $f(a) = \infty$ need to take a chart $1/w$ at $\infty$
$f$ analytic at $a \Rightarrow g(z) = \frac{1}{f(z)}$ is analytic in the neighbourhood of $a$ with $g(a) = 0$.
Writing $g(z) = (z - a)^r h(z)$ h analytic around $a$ and $h(a) \neq 0 r > 0$
$f(z) = (z - a)^{-r} \frac{1}{h(z)}$ with $\frac{1}{h(z)}$ holomorphic around $a$.
$\Rightarrow f$ has an isolated pole at $a$. Conversely if $f$ is meromorphic in the usual sense with a pole at $a \in U$ we can write
$f(z) = (z - a)^{-r} l(z)$ locally around $a$ for some holomorphic function $l$ with $l(a) \neq 0$.
$\Rightarrow g(z) = \frac{1}{f(z)} = (z - a)^r \frac{1}{l(z)}$ is analytic around $a$.
$f : U \to \mathbb{C}_\infty$ holomorphic. 

Example. (Related to Q 15 in Example Sheet 1)
$f(z) = z^3 - z$
$g(z) = \sqrt{z^3 - z}$

\[ \{ (z, w) : w^2 = f(z) \subset \mathbb{C} \} \]

$U = \mathbb{C} \setminus [-1, 0] \cap [1, \infty)$ Locally around every point in $\mathbb{C} \setminus \{0, \pm 1\}$, \exists precisely two functions $g$ such that $[g(z)]^2 = f(z)$
Claim: \exists $g$ holomorphic on $U$ s.t. $g^2(z) = f(z)$. We can do this by analytic continuation along paths.
Principle of Argument: $n(f \circ \gamma, 0) = n(\gamma, 0) + n(\gamma, 0) + n(\gamma, 1)$
$n(f \circ \gamma, 0) = 2n(\gamma, -1) \in 2\mathbb{Z}$

$n(f \circ \gamma, 0)$ even implies that as we analytically continue along a closed curve $\gamma$ we return exactly where we started so such a $g$ exists.
$R$: Riemann surface obtained from gluing.

$R$ is topologically a torus with four points removed.
3 The space of Germs and the Monodromy Theorem

First clear some topological preliminaries.

**Definition.** Suppose \( \pi : \tilde{X} \to X \) is a covering map of topological spaces and \( \gamma : [0, 1] \to X \) is a path. A lift of \( \gamma \) is a path \( \tilde{\gamma} : [0, 1] \to \tilde{X} \) such that \( \pi \circ \tilde{\gamma} = \gamma \).

**Proposition 3.1.** Uniqueness of Lifts Suppose \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) are lifts of \( \gamma \) with \( \tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) \) then \( \tilde{\gamma}_1 = \tilde{\gamma}_2 \).

**Proof.** Consider \( I = \{ t \in [0, 1] : \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t) \} \)

- \( I \) is open If \( \tau \in I \) choose open neighbourhood \( \tilde{N} \) of \( \tilde{\gamma}_1(\tau) = \tilde{\gamma}_2(\tau) \) such that \( \pi|_{\tilde{N}} \) is homeomorphic.
  By continuity there exists \( \delta > 0 \) : for \( t \in (\tau - \delta, \tau + \delta) \Rightarrow \tilde{\gamma}_1(t), \tilde{\gamma}_2(t) \in N \)
  \( \Rightarrow \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t) \) for all \( t \in (\tau - \delta, \tau + \delta) \)

- \( I \) is closed. If \( \tau \in I^C \) then \( \tilde{\gamma}_1 \tau \neq \tilde{\gamma}_2 \tau \)
  \( X \) Hausdorff \( \Rightarrow \exists \) open sets \( U_1, U_2 \subset \tilde{X} \) such that \( \tilde{\gamma}_1(\tau) \in U_1, \tilde{\gamma}_2(\tau) \in U_2, U_1 \cap U_2 = \emptyset \)
  Continuity \( \Rightarrow \exists \delta > 0 : |t - \tau| < \delta \Rightarrow \tilde{\gamma}_1(t) \in U_1, \tilde{\gamma}_2(t) \in U_2 \)

\( \square \)

**Definition.** A covering \( \pi : \tilde{X} \to X \) is called regular if every \( x \in X \) has an open neighbourhood \( U \) such that \( \pi^{-1}(U) = \bigsqcup \tilde{U}_\nu \)

where \( \tilde{U}_\nu \) are disjoint open subsets of \( \tilde{X} \) such that \( \pi|_{\tilde{U}_\nu} : \tilde{U}_\nu \to U \) is a homeomorphism for each \( \nu \).

**Example.** 1. Covering map

\[ \mathbb{C} \to \mathbb{C}^* \]

\[ z \mapsto \exp(z) \]

is regular but the restriction to \(-2\pi < \Im(z) < 2\pi \) is not.

2. \( \pi : \mathbb{C} \to \mathbb{C}/\Lambda, \Lambda \) lattice

**Proposition 3.2.** Let \( \pi : \tilde{X} \to X \) be a regular covering and \( \gamma \) a path in \( X, z \in \tilde{X} \) such that \( \pi(z) = \gamma(0) \). Then \( \exists \) a lift \( \tilde{\gamma} \) such that \( \tilde{\gamma}(0) = z \).

(Uniquely by (3.1))

**Proof.** Set \( I = \{ t \in [0, 1] : \exists \) a lift \( \tilde{\gamma} : [0, t] \to \tilde{X} \) with \( \tilde{\gamma}(0) = z \} \)

Clearly \( 0 \in I \) so \( I \neq \emptyset \).

Set \( \tau := \sup I \)
Claim: $\exists \delta > 0$ such that $[0, \tau + \delta] \cap [0, 1] \subset I$. This implies $\tau = 1$ and hence the result.

Proof of claim: Choose an open neighbourhood of $\gamma(\tau)$
with $\pi^{-1}(U) = \bigcup_{\nu} \tilde{U}_\nu$ as above

Continuity $\implies \exists \delta > 0: t \in [\tau - \delta, \tau + \delta] \cap [0, 1] \implies \gamma(t) \in U$

By definition of $\tau$, $\exists \tau_1 \in [\tau - \delta, \tau]: \gamma$ has a lift $\gamma(0) = z$

Now $\tilde{\gamma}(\tau_1) \in \pi^{-1}(U) = \bigcup_{\nu} \tilde{U}_\nu$.

$\exists! \nu$ s.t. $\tilde{\gamma}(\tau_1) \subset \tilde{U}_\nu$

Defining $\tilde{\gamma} := (\pi|_{\tilde{U}_\nu})^{-1} \circ \gamma$ on $[\tau, \tau + \delta] \cap [0, 1]$ extends $\tilde{\gamma}$ continuously to a lift on $[0, \tau + \delta] \cap [0, 1]$

$\blacksquare$

3.1 Homotopy of Lifts

Definition. Let $X$ be a topological space and $\alpha, \beta : [0, 1] \to X$ be paths with same initial and final points.

i.e. $\alpha(0) = \beta(0), \alpha(1) = \beta(1)$

We say that $\alpha, \beta$ are homotopic in $X$ if there exists a family $(\gamma_s)_{s \in [0,1]}$ of paths in $X$ s.t.

(i) $\gamma_0 = \alpha, \gamma_1 = \beta$

(ii) $\gamma_s(0) = \alpha(0) = \beta(0), \gamma_s(1) = \alpha(1) = \beta(1)$

(iii)

$$(t, s) \mapsto \gamma_s(t)$$

$[0, 1] \times [0, 1] \to X$

is continuous

Definition. $X$ is called simply connected if $X$ is path-connected and every closed path $\gamma : [0, 1] \to X$ is homotopic to the constant path $\gamma(0)$.

Theorem. Monodrony Theorem

Let $\pi : X \to X$ be a covering map and $\alpha, \beta$ paths in $X$. Assume

1. $\alpha, \beta$ homotopic in $X$

2. $\alpha, \beta$ have lifts $\tilde{\alpha}, \tilde{\beta}$ with $\tilde{\alpha}(0) = \tilde{\beta}(0)$

3. Every path $\gamma$ in $X$ with $\gamma(0) = \alpha(0) = \beta(0)$ have a lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{\alpha}(0) = \tilde{\beta}(0)$.

Then $\tilde{\alpha}, \tilde{\beta}$ homotopic in $X$ and in particular $\tilde{\alpha}(1) = \tilde{\beta}(1)$

Proof. (Non-examinable) See Beardon p.106-107

$\blacksquare$
Note: Regular⇒ (2) and (3) hold.
Let $G \subset \mathbb{C}$ be a domain.

**Definition.** Let $z \in G$, $(f, D), (g, E)$ be function elements in $G$. Write

$$(f, D) \sim_z (g, E)$$

if $z \in D \cap E, f = g$ on some open neighbourhood of $z$ in $D \cap E$.

This defines an equivalence relation and the equivalence classes containing $(f, D)$ is called the germ of $f$ at $z$ denoted by $[f]_z$.

$[f]_1 = [f]_2$ if $z_1 = z_2 = z$, $f = g$ on a neighbourhood of $z$.

$\mathcal{G} = \mathcal{G}(G) := \{\text{all germs } [f]_z \text{ over all } z \in G\}$ Let $(f, D)$ be a function element $\[f\]_D := \{[f]_z : z \in D\} \subset \mathcal{G}$

1. **Topology of $\mathcal{G}$**

| Open sets are unions of set of the form $[f]_D$ |
| Required to prove $U_1, U_2$ are open in $\mathcal{G} \Rightarrow U_1 \cap U_2$ is open |

**Proof.** Suppose $[f]_z \in U_1 \cap U_2$ since $U_1, U_2$ are open, $\exists(f_1, D_1)$ and $(f_2, D_2)$ s.t.

$[f]_z \in [f_1]_{D_1}$
$[f]_z \in [f_2]_{D_2}$

Choose representatives $(f, D)$ of the germ $[f]_z$ such that $z \in D \subset D_1 \cap D_2$ $[f]_z = [f_1]_z = [f_2]_z \Rightarrow \exists$ a neighbourhood of $z$: $f = f_1 = f_2$ on it. By the identity principle $f = f_1 = f_2$ on $D$.

Hence, $[f]_z \in [f]_D \subset U_1 \cap U_2$.

2. **Topology is Hausdorff**

Suppose $[f]_z \neq [g]_w$.

Choose domains $D, E$ such that $z \in D, w \in E$ $f$ analytic on $D$, $g$ analytic on $E$.

If $z \neq w$ choose $D, E$ disjoint and $[f]_D \cap [g]_E = \emptyset$.

If $z = w$ choose $D$ and $E$ to coincide and claim $[f]_D \cap [g]_E = \emptyset$.

Otherwise $[h]_z \in [f]_D \cap [g]_E$ for $s \in D$.

$\Rightarrow h = f = g$ is a neighbourhood of $S$.

$\Rightarrow f = g$ on $D$ by the identity principle.

$\Rightarrow [f]_z \neq [g]_z$ #Absurd

3. **Projection map**

$\pi : \mathcal{G} \to G$
$[f]_z \mapsto z$

- $\pi$ is continuous
  Given $V \subset G$ open, note $\pi^{-1}(V) = [f]_D$

- $\pi|[f]_D : [f]_D \to D$ is a homeomorphism.
  Certainly bijective and $\pi|[f]_D$ is continuous.
  $\pi|[f]_D$ is an open map.

Take $U$ open in $[f]_D$. 

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20
4. Atlas on $\mathcal{G}$

On each connected component $\pi$ is a covering map so we use Lemma 2.2 to give each connected component the structure of a Riemann surface for which $\pi$ is analytic.

Charts are simply $(U, \phi)$, $U = [f]_D$

5. Evaluation Map

$E : \mathcal{G} \to \mathbb{C}$

$[f]_z \mapsto f(z)$

This is analytic.

$E\phi^{-1}(z) = E([f]_z) = f(z)$ holomorphic. $(f, D), z \in D$

**Theorem 3.3.** Let $(f, D)$ and $(g, E)$ be function elements in $G$ and $\gamma : [0, 1] \to G$ be a path with $\gamma(0) \in D, \gamma(1) \in E$.

Then $(g, E)$ is an analytic continuation of $(f, D)$ along $\gamma$ if $\exists$ a path $\tilde{\gamma} : [0, 1] \to \mathcal{G}$ such that $\pi\tilde{\gamma} = \gamma$ and

$\tilde{\gamma}(0) = [f]_{\gamma(0)}$

$\tilde{\gamma}(1) = [g]_{\gamma(1)}$

**Proof.** Assume $(f, D) \sim_{\gamma} (g, E)$

$\Rightarrow \exists (f_j, D_j)_{j=1}^n$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ with

$(f, D) = (f_1, D_1) \sim (f_2, D_2) \sim \ldots \sim (f_n, D_n) = (g, E)$

and $\gamma([t_{j-1}, t_j]) \subset D_j$ for $j = 1, \ldots, n$.

Define $\tilde{\gamma}(t) := [f_j]_{\gamma(t)}$ for $t_{j-1} \leq t \leq t_j$

Note this is well-defined since $f_j = f_{j-1}$ on $D_{j-1} \cap D_j$ (Thus take care of endpoints.)

Obviously $\pi\tilde{\gamma} = \gamma$

Required to prove $\tilde{\gamma}$ is continuous, $\tilde{\gamma} : [0, 1] \to \mathcal{G}$

Consider a basic open set $[h]_\Delta$ and suppose $\tilde{\gamma}(\tau) \in [h]_\Delta$

Without loss of generality, we may assume that $\tau \in [t_{j-1}, t_j]$ so

$\tilde{\gamma}(\tau) = [f_j]_{\gamma(\tau)} \in [h]_\Delta$

i.e. $\gamma(\tau) \subset \Delta$ and $f_j = h$ is some open neighbourhood $N$ of $\gamma(\tau)$ in $\Delta \cap D_j$

By continuity of $\gamma$, there is a $\delta > 0$ s.t. if $|t - \tau| < \delta \Rightarrow \gamma(t) \in N$

Then $\tilde{\gamma}(t) = [f_j]_{\gamma} = [h]_{\gamma(t)} \in [h]_\Delta$
\[ \Rightarrow \hat{\gamma} \text{ is continuous.} \]
\[ (\Leftarrow) \text{ Assume } \exists \hat{\gamma} \to \mathcal{G} \text{ is a path s.t. } \hat{\gamma}(0) = [f]_{\gamma(0)}, \hat{\gamma}(1) = [g]_{\gamma(0)} \text{ and } \pi \hat{\gamma} = \gamma \]
\[ \textbf{Claim:} \text{ We can find basic open sets } [f]_{D_1}, \ldots, [f]_{D_n} \text{ in } \mathcal{G} \text{ for some } n \text{ and } 0 = t_0 < t_1 < \ldots < t_n = 1 \text{ s.t. } \hat{\gamma}(\{[t - 1], t_j\}) \subset [f_j]D_j \text{ for each } j. \]
\[ \textbf{Proof of claim:} \text{ Note that by continuity of } \hat{\gamma} \text{ for each } t \in [0, 1] \text{ we can find a basic open set } [f]_{D_1} \ni \hat{\gamma}(t) \text{ in } \mathcal{G} \text{ and an open interval } I(t) \nexists t \text{ such that } \hat{\gamma}(I(t) \cap [0, 1]) \subset [f_i]D_i \]
The intervals \( I_k \) cover \( [0, 1] \) hence by compactness we can find a finite subcover, which we can reorder to obtain intervals \( I_k = [a_k, b_k] \) with \( a_k < b_k \) for all \( k = 1, \ldots, n. \)
Choosing \( t_k \in (a_{k+1}, b_k) \) we can arrange \( f_k, \hat{\gamma}(\{[t_j-1], t_j\}) \subset [f_j]D_j \)
\[ \textbf{Adjustments: WLOG may assume that} \]
1. Each \( D_j \) to be a disc.
2. Also, since \( \hat{\gamma}(0) = [f]_{\gamma(0)} \) and \( \hat{\gamma}(1) = [g]_{\gamma(1)} \)
   We may assume that \( D_1 \subset D, D_n \subset E \) with \( f = f_1 \) in \( D_1 \) and \( g = f_n \) in \( D_n. \)
   For each \( 1 \leq j \leq n - 1, \hat{\gamma}(t_j) \in [f_j]D_j \cap [f_j+1]D_{j+1} \), hence \( f_j = f_j + 1 \) on a range of \( \gamma(t_j) \) and hence in all of \( D_j \cap D_{j+1} \)
   (Identity Principle on intersection \( D_j \cap D_{j+1} \) is connected since \( D_i \) are disc(s).
   Thus, \( (f, D) \sim (f, D_1) = (f_1, D_1) \sim \ldots \sim (f_n, D_n) = (g, D_n) \sim (g, E) \)
   And we have \( \gamma(\{[t_j-1], t_j\}) \subset \pi \hat{\gamma}(\{[t_j-1], t_j\}) \subset \pi([f_j]D_j) = D_j \)
   (We can always enlarge the dissection \( 0 < \frac{t_j}{2} < t_j < \ldots < t_{n-1} < \frac{t_{n-1}+1}{2} < 1 \)
to account for his first and last continuation.)

\[ \Box \]

\[ \textbf{Corollary 3.4.} \text{ Let } \mathcal{F} \text{ be a complete analytic function in } G \text{ and put } \mathcal{G}_\mathcal{F} = \bigcup_{(f, D) \in \mathcal{F}} [f]D. \text{ Then } \mathcal{G}_\mathcal{F} \text{ is a connected component of } \mathcal{G}. \]

\[ \textbf{Proof.} \text{ (3.3) } \Rightarrow \mathcal{G}_\mathcal{F} \text{ is path-connected, hence connected.} \]
\[ \mathcal{G}_\mathcal{F} \text{ is open by definition.} \]
\[ \text{Also } \mathcal{G}_\mathcal{F} = \mathcal{G} \cap_{\mathcal{F} \sim \mathcal{F}} \mathcal{G}_\mathcal{F}. \]
\[ \text{Hence } \mathcal{G}_\mathcal{F} \text{ is a connected component.} \]

\[ \Box \]

\[ \textbf{Summary:} \text{ } G \subset \mathbb{C}, \mathcal{F} \text{ complete analytic function then we have constructed a Riemann Surface } \mathcal{G}_\mathcal{F} \text{ and a diagram.} \]
\[ \pi : \mathcal{G}_\mathcal{F} \to G \text{ projection map which is analytic covering.} \]
\[ \mathcal{G}_\mathcal{F} \text{ is the Riemann Surface associated with } \mathcal{F}. \]
**Theorem 3.5** (Uniqueness of Continuation along paths). If \( (g, E) \) and \( (h, E) \) are analytic continuations of \( (f, D) \) along some path \( \gamma \) in \( G \) then \( g = h \) in \( E \).

**Proof.** \( (3.3) \Rightarrow \exists \tilde{\gamma}, \tilde{\gamma}^* \) s.t. \( \tilde{\gamma}(0) = \tilde{\gamma}^*(0) = [f]_{\gamma(0)} \) in \( \mathcal{F} \) and

\[
\begin{align*}
\tilde{\gamma}(1) &= [g]_{\gamma(1)} \\
\tilde{\gamma}^*(1) &= [h]_{\gamma(1)}
\end{align*}
\]

Uniqueness of lifts \( (3.1) \Rightarrow \tilde{\gamma}(1) = \tilde{\gamma}^*(1) \)

\( \Rightarrow g = h \) in a neighbourhood of \( \gamma(1) \)

\( \Rightarrow g = h \) on \( E \) by the identity principle.

**Theorem 3.6** (Classical Monodromy Theorem). Suppose \( (f, D) \) can be continued along all paths in \( G \) which start in \( D \). If \( (g, E) \) and \( (h, E) \) are continuations of \( (f, D) \) along paths \( \alpha, \beta \) which are homotopic in \( G \), then \( g = h \) on \( E \).

**Proof.** \( (3.3) \Rightarrow \exists \text{ lifts } \tilde{\alpha}, \tilde{\beta} \) s.t.

\[
\begin{align*}
\tilde{\alpha}(0) &= [f]_{\alpha(0)} , \tilde{\beta}(0) = [f]_{\beta(0)} \\
\tilde{\alpha}(1) &= [f]_{\alpha(1)} , \tilde{\beta}(1) = [f]_{\beta(1)}
\end{align*}
\]

Since \( \alpha(0) = \beta(0) \) we have \( \tilde{\alpha}(0) = \tilde{\beta}(0) \).

So by Monodromy Theorem (Topological) now lifts \( \tilde{\alpha}, \tilde{\beta} \) are homotopic and in particular, \( \tilde{\alpha}(1) = \tilde{\beta}(1) \) Hence as before \( g = h \) on \( E \).

**Corollary 3.7.** Suppose \( G \subset \mathbb{C} \) is simply-connected and \( (f, D) \) is a function element in \( G \) which can be analytically continued along all paths in \( G \) which start in \( D \). Then, \( \exists \) a single-valued analytic continuation \( \tilde{f} \) of \( (f, D) \) to all \( G \).

**Proof.** If \( (g, E) \) and \( (h, E) \) are two analytic continuations of \( (f, D) \) then the two analytic continuations of each other along some closed path \( \gamma \) in \( G \).

Since \( G \) is simply-connected, \( \gamma \) is homotopic to a point, hence \( (3.6) \Rightarrow g = h \) on \( E \).

Deforming \( \tilde{f} = g \) on \( E \)(for all \( (g, E) \in \mathcal{F} \)) gives an analytic extension of \( (f, D) \) to all of \( G \).
4 Multiplicities and Degrees

$f : R \rightarrow S$ with $R, S$ compact Riemann surfaces, $f$ non-constant.
Given $z \in U \subset \mathbb{C}$ (domain) and a non-constant holomorphic function $f$, we can write locally
\[ f(w) = f(z) + (w - z)^{m_r(z)}f_1(w) \]
where $f_1(z) \neq 0$
We call $m_f$ the multiplicity or valency of $f$ at $z$.

Lemma 4.1. Suppose $g$ and $h$ are analytic and non-constant on domains in $\mathbb{C}$ with $\text{image}(h) \subset \text{image}(g)$ then,
\[ m_{g \circ h}(z) = m_g(h(z))m_h(z) \]

Proof. 1. $h(w) = h(z) + (w - z)^{m_h(z)}h_1(w)$ locally with $h_1(z) \neq 0$
2. $g(\eta) = g(\eta_0) + (\eta - \eta_0)^{m_g(\eta)}g_1(\eta)$ locally with $g_1(\eta) \neq 0$ and $\eta_0 = h(z)$.
Substitute (1) into (2).
\[ g(h(w)) = g(h(z)) + (w - z)^{m_h(z)m_g(\eta_0)}(h_1(w))^{m_g(\eta_0)}g_1(h(w)) \]
locally with $g_1(h(z)) = g(\eta_0) \neq 0$ and $h_1(z) \neq 0$
Consider $f : R \rightarrow S$ analytic and non-constant and $z \in \mathbb{R}$.
Take charts $(U, \phi)$ around $z$ and $(V, \psi)$ around $f(z)$ and define $m_f(z) = m_{\psi \circ \phi^{-1}}(\phi(z))$
This does not depend on the choice of charts.
If we choose $(\tilde{U}, \tilde{\phi})$, $(\tilde{V}, \tilde{\psi})$ instead then,
\[ \tilde{\psi} \tilde{f} \tilde{\phi}^{-1} = \tilde{\psi} \psi^{-1} \psi \tilde{f} \phi^{-1} \phi \tilde{\phi}^{-1} \quad \text{local conformal equivalence} \]
Since $m_{\tilde{\psi}^{-1}}(\psi(f(z))) = 1$ and $m_{\tilde{\phi}^{-1}}(\phi(f(z))) = 1$
(4.1) $\implies m_{\tilde{\psi}^{-1}, \tilde{\phi}^{-1}}(\phi(z)) = m_{\psi, \phi^{-1}}(\phi(z))$

Remark. 1. $m_f(z) > 1$ only at isolated points of $R$. ($f$ non-constant.)
Clear from the plane case given by vanishing of $f'$.
2. Points $z \in \mathbb{R}$ s.t. $m_f(z) > 1$ are called "Ramification Points" and $m_f(z)$
is called the ramification index. The image of a ramification is called a
branch point.
Example. \( p(z) = \sum_{0}^{d} a_k z^k \) polynomial of degree \( d \), \( a_d \neq 0 \).

Define \( p(\infty) = \infty \). Then \( p \) becomes a holomorphic map \( \mathbb{C}_\infty \to \mathbb{C}_\infty \).

Take charts \( \phi(w) = \psi(w) = \frac{1}{w}, \frac{1}{w} \) on \( \mathbb{C}_\infty \setminus \{0\} \).

\[
\psi p \phi^{-1}(z) = \frac{1}{p(1/z)} = \sum_{0}^{d} a_k z^{d-k} = z^d g(z) \text{ with } g(0) \neq 0
\]

\( \Rightarrow m_p(\infty) = d \)

**Theorem 4.2** (Valency Theorem). Suppose \( f : R \to S \) is a non-constant holomorphic map with \( R \) compact (\( \Rightarrow f(R) = S \) and \( S \) compact.) Then \( \exists \) an integer \( n \geq 1 \) s.t. \( f \) is an \( n \) to 1 map of \( R \) onto \( S \) counting multiplicities.

\[
n = \sum_{z \in f^{-1}(w)} m_f(z) \quad \forall w \in S
\]

**Definition.** \( n \) is called the **degree** of \( f \), also denoted \( n(f), \text{deg}(f) \).

**Proof.** \( f^{-1}(w) \) is finite. Note \( f^{-1}(w) \) is compact and discrete.(Identity Principle)

Define \( n(w) := \sum_{z \in f^{-1}(w)} m_f(z) \) for \( w \in S \).

Our aim is to prove that \( w \mapsto n(w) \) is constant.

Take \( w_0 \in S \) and write \( f^{-1}(w_0) = \{z_1, ..., z_q\} \)

For any point \( z_0 \in R \), we know that \( \exists \) charts

\[
\psi : N \mapsto \mathbb{C} \\
\theta : N \mapsto \mathbb{C}
\]

with \( f(n) \subset V \) and \( \theta f \psi^{-1} \) is just \( z \mapsto z^{m_p(z_0)} \)

Hence \( \exists \) disjoint neighbourhoods \( N_1, ..., N_q \) of \( z_1, ..., z_q \) s.t. \( f \) is an \( m_f(z_j) \) to 1 map. (Counting multiplicities) of \( N_j \) onto \( f(N_j) \) for each \( j = 1, ..., q \).

Include image

**Claim.** There exists neighbourhood \( M \) of \( w_0 \) s.t. \( f(R \setminus \bigcup_{1}^{q} N_j) \cap M = \phi \)

**Proof of claim.** \( R \setminus \bigcup_{1}^{q} N_j \) closed in \( R \).

Hence compact \( \Rightarrow f(R \setminus \bigcup_{1}^{q} N_j) \) is compact hence closed.

Then \( M = S \setminus f(R \setminus \bigcup_{1}^{q} N_j) \) is open containing \( w_0 \) and \( M \cap f(R \setminus \bigcup_{1}^{j} N_j) = \phi \)

\( \square \)

**Corollary 4.3** (Fundamental Theorem of Algebra, one more time!). Let \( P \) be a polynomial of degree \( d \), then \( P \) has \( d \) zeroes counted with multiplicities.

25
Proof 1. We have seen that $P$ gives rise to $P : \mathbb{C}_\infty \to \mathbb{C}_\infty$ holomorphic with $P(\infty) = \infty$ and $m_P(\infty) = d \Rightarrow \# \text{ of zeroes of } P = \sum_{z \in P^{-1}(0)} m_P(z) = d \square$

Proof 2. By (2.5) $p$ is onto. Hence there exists a zero of $p$. Use remainder theorem to get $d$ zeroes. \square

4.1 Rational Maps

Proposition 4.4. $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ holomorphic and non-constant
\[ \Leftrightarrow f(z) = \frac{c(z - \alpha_1)...(z - \alpha_n)}{(z - \beta_1)...(z - \beta_2)} \text{ a non-constant rational function in } \mathbb{C}. \]

Proof. ($\Leftarrow$) $f(z)$ and $f(\frac{1}{z})$ are meromorphic and so define a holomorphic map of $\mathbb{C}_\infty$.
($\Rightarrow$) W.l.o.g assume $f(\infty) \in \mathbb{C}$. Otherwise consider \( \frac{1}{f} \). Then $f^{-1}(\infty)$ is a finite subset of $\mathbb{C}$ say \( \{z_1, ..., z_q\} \) and $f$ is meromorphic in $\mathbb{C}$ as each $z_j$ is a pole. Near $z_j$, $f$ has the form,
\[
 f(z) = \sum_{-k_j}^{\infty} a_i(z - z_j)^i
\]
and set $Q_j = \sum_{-k_j}^{1} a_i(z - z_j)^i$

Consider $\tilde{f} = f - (Q_1 + ... + Q_q)$. $\tilde{f}$ has now only removable singularity at $z_1, ..., z_q$. Hence $\tilde{f}$ gives rise to a holomorphic map $\tilde{f} : \mathbb{C}_\infty \to \mathbb{C}_\infty$. By (2.6) $\tilde{f}$ is constant and thus $f = Q_1 + ... + Q_q + \text{constant.}$ \square

Remark. $f(\infty) \in \mathbb{C} \Leftrightarrow n \leq m$ and in this case $\deg(f) = m = \# \text{ of poles counting multiplicities}$.

(Example Sheet 2) $\deg(f) = \max\{m,n\}$
If $f$ is an analytic isomorphism, $\deg(f) = 1 \Rightarrow m = n = 1$.
Hence $f(z) = az + b \bar{c}z + d$ with $ad - bc \neq 0$, a Möbius map.
So $\text{Aut}(\mathbb{C}_\infty) =$ Möbius group.

4.2 Quick review of triangulations and Euler characteristic (from IB Geometry)

$S$ compact surface.
Topological triangle on $S$=homeomorphic image of a closed triangle $T \subset \mathbb{R}^2$.

Definition. A topological triangulation consists of a finite collection of topological triangles whose union is all of $S$ s.t. (i) Two triangles are either disjoint or their intersection is a common vertex or common edge.
\( F = \# \) of triangles  
\( V = \# \) of vertices  
\( E = \# \) of edges  
\[ e := F - E + V \]

Euler-number of triangulation

**Fact.** Fact 1 One can always find a triangulation.

**Fact.** Fact 2 \( e \) does not depend on the chosen triangulation; it is a topological invariant \( \chi(S) \).

**Example.** 1. \( S^2 \)
\[
F = 8 \\
E = 12 \\
V = 6 \\
\chi(S^2) = 2
\]

2. Torus \( T^2 \) \( F = 18, E = 27, V = 9 \)
\[
\chi(T^2) = 0
\]

**Fact.** A compact Riemann Surface \( R \) is topologically a sphere with \( g \) handles. e.g. \( g = 0 \).

\[
\chi(R) = 2 - 2g \quad g \text{ genus of } R.
\]

\( f : R \to S \) be a non-constant holomorphic map between compact Riemann Surfaces. Given \( p \in R \), let \( e_p \) be the ramification index \( m_f(p) \). Recall \( e_p > 1 \Leftrightarrow p \) is a ramification point and the set of these points is actually finite. In particular, the following sum

\[
\sum_{p \in R} e_p - 1 \quad \text{is finite.}
\]

**Theorem 4.5** (Riemann Hurwitz). If \( f \) has degree \( n \), then

\[
\chi(R) = n \chi(S) - \sum_{p \in R} (e_p - 1)
\]

**Proof.** Let \( Q_1, ..., Q_r \) denote the branch points of \( f \). Suppose we have a triangulation on \( S \) by subdividing it we can assume that the \( Q_i \) are vertices.

The proof of the valency theorem theorem (4.2) gives us an open cover \( U_1, ..., U_r, U_{r+1}, ..., U_m \) of \( S \) such that

(i) If \( j > r \), \( f^{-1}(U_j) \) has a disjoint union of open subset of \( V \) for which \( f|_V : V \to U \) is a conformal equivalence so \( U \) contains no branch points.

(ii) If \( j \leq r \) \( Q_j \in U_j \) and for each point \( p \in f^{-1}(Q_j) \) there exists a unique component \( V \) of \( f^{-1}(U_j) \) so that \( f|_V : V \to U_j \) is an \( e_p \) to 1 map over \( U_j \setminus \{Q_j\} \) given in terms of suitable charts by \( z \mapsto z^{e_p} \).

Now we ??? our triangulation by means of Euclidean triangles as follows.
...to ensure that each topological triangle is centred in some $U_j$. 

$T$ then lifts to a triangle on $R$ if no vertex of $T$ is a branch point, these triangles are disjoint; either $T \subset U_j$ if $j > r$ (when it is obvious) or $T \subset U_j$ if $j < r$ and still okay. If $T$ has a vertex at a branch point $Q_i$, it still lifts to $n$ triangles of $R$ but there will be $e_p$ vertices in common at each ramification point $p \in f^{-1}(Q_j)$. We therefore get at triangulation on $R$ corresponding to the one in $S$ s.t. $F, F', V, V', E, E'$ denoted the # of faces, vertices and edges respectively on $S$ and $R$ and we see that $F' = uF, E' = uE, V' = nV - \sum_{p \in R} e_p - 1$. Hence $\chi(R) = n\chi(S) - \sum_{p \in R} e_p - 1$. \hfill $\square$

**Trivial Remark.**

$$2 - 2g(R) = n(2 - 2g(S)) - \sum_{p \in R} e_p - 1$$

$$\implies g(R) \geq g(S)$$

**Example.** $R \subset \mathbb{C} \times \mathbb{C}$ $R = \{(z, w)|w^2 = f(z)\}$ where $f(z) = z^3 - z$

$$\pi : R \rightarrow \mathbb{C}$$

$$(0, 0) = \pi^{-1}(0)$$

$$(1, 0) = \pi^{-1}(1)$$

$$(-1, 0) = \pi^{-1}(-1)$$

$$g : R \rightarrow \mathbb{C}$$

$$(z, w) \mapsto w$$

$g$ around these 3 points is an inverse. $g^{-1}(w) = (f^{-1}(w^2), w)$

Look at $\pi$ around $(0, 0)$ and $(\pm 1, 0)$

$$\pi g^{-1}(w) = f^{-1}(w^2)$$

$$\frac{d\pi g^{-1}}{dw} = (f^{-1})'(w^2) \cdot 2w$$

Assume $R$ can be compactified. We can add a point $c$ s.t. $\bar{R} = R \cup \{c\}$ is a compact Riemann Surface and $\pi(c) = \infty$ so that $\pi : \bar{R} \rightarrow \bar{C}_\infty$ is holomorphic. $deg(\pi) = 2$, branch points $0, \pm 1$ ramification points $(0, 0), (\pm 1, 0)$ with $e_p = 2$.

Since $\pi^{-1}(\infty) = c$ then $\infty$ is also a branch point and $e_c = 2$.

Riemann Hurwitz $\chi(\bar{R}) = 2\chi(\mathbb{C}_\infty) - 4 = 2 \times 2 - 4 = 0 \implies g(\bar{R}) = 1$

Compactification $\bar{R}$ is a torus.

One could push this further and consider $f$ a polynomial with $2g + 1$ simple roots to obtain a Riemann Surface $\bar{R}$ with genus $g$. 

28
5 Meromorphic Periodic Functions

$f : \mathbb{C} \to \mathbb{C}_\infty$ analytic and non-constant. We say $w$ is a period for $f$ if $f(z+w) = f(z)$ for all $z \in \mathbb{C}$.

The set of period $\Omega$ satisfies:

1. $\Omega$ is an additive subgroup of $\mathbb{C}$.
2. $\Omega$ contains only isolated points.

Example sheet #3 Q1 $\Rightarrow$ Only 3 possibilities for $\Omega$.

1. $\Omega = \{0\}$ non-periodic
2. $\Omega = \mathbb{Z}w_1 \neq 0$ simply periodic
3. $\Omega = \mathbb{Z}w_1 + \mathbb{Z}w_2 \neq 0$, $\frac{w_1}{w_2} \notin \mathbb{R}$ Doubly periodic or Elliptic.

5.1 Simply Periodic Functions

Without loss of generality, assume $\Omega = \mathbb{Z}$. i.e. $f(z+1) = f(z)$ for all $z \in \mathbb{C}$

**Proposition 5.1.** Define $p : \mathbb{C} \to \mathbb{C}^*$ by $p(z) = e^{2\pi iz}$. If $f$ is meromorphic in $\mathbb{C}$ with periods $\mathbb{Z}$ then there exists a unique $\tilde{f}$ meromorphic on $\mathbb{C}^*$ s.t. $f = \tilde{f} \circ p$

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \mathbb{C}_\infty \\
\downarrow p & \nearrow \tilde{f} & \\
\mathbb{C}^* & & \\
\end{array}
\]

**Proof.** Define $\tilde{f}(e^{2\pi iz}) = f(z)$.

Well defined: If $e^{2\pi iz_1} = e^{2\pi iz_2} \Leftrightarrow z_1 - z_2 \in \mathbb{Z} \Leftrightarrow f(z_1) = f(z_2)$ Clearly $f = \tilde{f} \circ p$ and $\tilde{f}$ is unique.

If $V$ is open in $\mathbb{C}_\infty$ then $p^{-1}\tilde{f}^{-1}V = f^{-1}V$ is open in $\mathbb{C}$

$\Rightarrow f^{-1}V$ open

$\tilde{f}$ meromorphic locally with $\tilde{f}(w) = f \left( \frac{\log(w)}{2\pi i} \right)$

**Corollary 5.2.** Suppose $f$ has no poles on the strip $S = \{z : \alpha < Im(z) < \beta\}$ Then $f(z) = \sum_{-\infty}^{\infty} a_k e^{2\pi ikz}$ is locally uniformly convergent on $S$.

**Proof.** $\tilde{f}$ is analytic on the annulus, $p(S) = \{w : e^{-2\pi \beta} < |w| < e^{-2\pi \alpha}\}$ Lau-

\[
\Rightarrow \tilde{f}(w) = \sum_{-\infty}^{\infty} a_k w^k 
\]

(Fourier expansion of simply periodic functions)

29
5.2 Doubly Periodic Functions

Let $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ be a lattice in $\mathbb{C}$. A meromorphic function $f$ on $\mathbb{C}$ with period $\Lambda$ is called elliptic.

These functions form a field.

An elliptic function is determined by its value on any parallelogram $P = \{z + t_1 w_1 + t_2 w_2 : t_1, t_2 \in [0, 1)\}$

If it has no poles $\Rightarrow$ bounded in $\mathbb{C}$ $\Rightarrow$ constant by Liouville’s theorem.

Now we look at the complex torus $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$.

**Proposition 5.3.** If $f$ is an elliptic function with period $\Omega$ then there exists a unique meromorphic map $\tilde{f}$ on $\mathbb{C}/\Lambda$ s.t. $f = \tilde{f} \circ \pi$

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{f} & \mathbb{C}_\infty \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{C}/\Lambda & \xrightarrow{\tilde{f}} & \mathbb{C}_\infty
\end{array}
$$

**Proof.** Same as Proposition 5.1(Exercise). \qed

**Corollary 5.4.** The degree of $\tilde{f}$ is $n \geq 2$.

**Proof 1.** If $\deg(\tilde{f}) = 1$ then $\tilde{f} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ is a conformal equivalence. But this is absurd since they have different genus. ($\mathbb{C}/\Lambda$ and $\mathbb{C}_\infty$) \qed

**Proof 2.** Consider a period parallelogram $P$ such that $\partial P$ contains no poles of $f$.

$$
\sum_{z \in P} \text{Res}_z f = \frac{1}{2\pi i} \int_{\partial P} f(z) dz = 0
$$

$\Rightarrow f$ cannot just have a simple pole. \qed

5.3 Weierstrass $\wp$ function

**Definition.** For a given lattice $\Lambda$ define $\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{z \in \Lambda \setminus \{0\}} \{\frac{1}{(z-w)^2}\} - \{\frac{1}{w^2}\}$

Convergence?

**Lemma.** \[ \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{|w|^t} < \infty \iff t > 2 \]

**Proof.** The function $(t_1, t_2) \mapsto t_1 w_1 + t_2 w_2$ is continuous and non-zero on the compact set \{$(t_1, t_2) : |t_1| + |t_2| = 1$\}

And hence its modulus is bound above and below by constants $C, c > 0$.

Set $t_1 = \frac{k}{|k| + |l|}$ and $t_2 = \frac{l}{|k| + |l|}$

$(k, l) \in \mathbb{Z}^2 \setminus \{0, 0\}$

$\Rightarrow c(|k| + |l|) \leq |k w_1 + l w_2| \leq C(|k| + |l|)$

30
Hence,
\[ \sum_{(k,l) \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{|w_1^k + w_2^l|^4} < \infty \]
\[ \Leftrightarrow \sum_{q=1}^{\infty} \sum_{|k|+|l|=1} \frac{1}{q^2} < \infty \]
\[ \Leftrightarrow \sum_{q=1}^{\infty} \frac{4}{q^2 - 1} < \infty \Leftrightarrow t > 2 \]

\[ \square \]

**Theorem 5.5.** \( \wp \) is elliptic with periods \( \Lambda \) Moreover, \( \wp \) is an even function. \( \wp \) is 2 to 1 on a fundamental/period parallelogram.

**Proof.** Let \( |z| < r \) and \( |w| > 2r \).

\[
\left| \frac{1}{(z - w)^2} - \frac{1}{w^2} \right| = \left| \frac{2wz - z^2}{(w - z)^2w^2} \right| \leq \frac{2r|w| + r^2}{|w|^4/4},
\]

|z| < r, and \( |w-z| > \frac{|w|}{2} \)

So then, \( \wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda, 0 < |w| < 2r} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right) + \sum_{|w| \geq 2r} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right) \)

converges uniformly for \( |z| < r \)

So indeed \( \wp \) is meromorphic.

Tivially \( \wp \) is even.

By uniform convergence, we can differentiate termwise to see,

\[ \wp'(z) = \sum_{w \in \Lambda} \frac{(-2)}{(z - w)^2} \]

so clearly we have,

\[ \wp'(z + w) = \wp'(z) \]

for all \( w \in \Lambda \)

So for fixed \( w \in \Lambda \), \( \wp(z + w) - \wp(z) = c \) is a constant (independent of \( z \)). Replace \( z \) by \( -z - w \) in the last line.

\[ \wp(z + w) - \wp(z) = c = \wp(-z) = -\wp(-z - w) = -c \]

\( \Rightarrow c = 0 \) So \( \Lambda \) is contained in the periods of \( \wp \) and since \( \wp \) only has roots on \( \Lambda \),

these are all the periods.

In a fundamental parallelogram, \( \wp \) has a double pole.

So by the valency theorem, this implies that \( \wp \) defines a 2 to 1 map \( \mathbb{C}/\Lambda \rightarrow \mathbb{C}_\infty \)

\[ \square \]
Remark. \( \wp \) is characterised by having periods \( \Lambda \), poles only on \( \Lambda \) and \( \wp(z) - \frac{1}{z^2} \to 0 \) as \( z \to 0 \)

Why? Let \( f \) be another meromorphic function. Then \( f - \wp \) is periodic with at worst simple poles...It’s constant \((\text{deg} \geq 2)\)

Now considering \( f(z) - \wp(z) = \left( f(z) - \frac{1}{z^2} \right) - \left( \wp(z) - \frac{1}{z^2} \right) \)

We see \( f - \wp \equiv 0 \).

The function \( \wp' \) has triple poles on \( \Lambda \) and valency 3. It is clearly odd and \( \wp' \left( \frac{w_1}{z} \right) = \frac{1}{z^2} \)

\( \Rightarrow \frac{w_1}{z} \) is a zero of \( \wp' \)

Similarly, \( \frac{w_2}{z} \) and \( \frac{w_1 + w_2}{z} \) are zeroes of \( \wp'(z) \).

Therefore the analytic map \( \wp : \mathbb{C}/\Lambda \to \mathbb{C}_\infty \) is ramified at the points 0, \( \frac{w_1}{z} \), \( \frac{w_2}{z} \), \( \frac{w_1 + w_2}{z} \) and it has branch points \( \infty, \wp \left( \frac{w_1}{2} = e_1 \right), \wp \left( \frac{w_2}{2} = e_3 \right), \wp \left( \frac{w_1 + w_2}{2} = e_2 \right) \)

Check: 4 branch points is compatible with degree-genus formula for \( \text{Riemann-Hurwitz} \)

Note: The points \( e_1, e_2 \) and \( e_3 \) are all distinct because otherwise \( \wp \) would have valency at least 4.

Proposition 5.6. There is an algebraic relation \( (\wp'(z))^2 = 4 (\wp(z))^3 - g_2 \wp(z) - g_3 \)

for constant \( g_2, g_3 \in \mathbb{C} \) depending only on \( \Lambda \).

Proof. Using \( \wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \), the Laurent expansion of \( \wp(z) \) over 0 has

\( \wp(z) = \frac{1}{z^2} + az^2 + O(z^4) \)

So \( \wp'(z) = -\frac{2}{z^3} + 2az + O(z^3) \)

\( (\wp'(z))^2 - 4(\wp(z))^3 = \frac{-g_2}{z^2} + (\text{locally holomorphic function}) \)

\( \Rightarrow \wp'(z) - 4\wp(z) + g_2\wp(z) \) is constant by periodicity. (Liouville’s theorem)
Remark. Since $\wp'(z) = 0$ if $z = \frac{w_1}{2} + \frac{w_2}{2}$, we find that the cubic equation,

$$4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

In particular, note that $e_1 + e_2 + e_3 = 0$

Remark. Another interpretation of Proposition 5.6.
Consider the map $(\wp(z), \wp'(z))$ as defining an analytic map.

$$F : (\mathbb{C}/\Lambda) \setminus \{0\} \to \mathbb{C}^2$$

The image of $F$ lies on the cubic curve, $y^2 = 4x^3 - g_2x - g_3$.
This can be compactified (cf. Example Sheet 1 Q15) to a curve in $\mathbb{P}^3(\mathbb{C})$ defined by the homogeneous equation

$$y^2z = 4x^3 - g_2xz^2 - g_3z^3$$

$[x : y : z]$ coordinate on $\mathbb{P}^3(z)$

Last Task: We want to understand all elliptic functions on $\mathbb{C}/\Lambda$

**Theorem 5.7.** Let $f$ be an elliptic function with period $\Lambda$. Then there are rational functions on $Q_1, Q_2$.

$$f = Q_1(\wp) + \wp'Q_2(\wp)$$

In particular, if $f$ is even then you can take $Q_2 \equiv 0$

**Proof.** We first treat the case when $f$ is assumed to be even. Let $E = \left\{ t \in \mathbb{C} | z \in \frac{1}{2}\Lambda \text{ or } f'(z) = 0 \right\}$

Then $f(E)$ is finite, so we can choose distinct, $d \in \mathbb{C} \setminus f(E)$.

Let $g(z) = \frac{f(z) - c}{f(z) - d}$ so at poles of $f$, $g(z) \to 1$

Then $g$ is elliptic and its zeroes, (where $f(z) = c$) and poles $(f(z) = d)$ are all simple as $f'(z) = 0$ at each point and $g$ is even since $f$ is.

Let us list the zeroes and poles of $g$ as $\{a_1, ..., a_m, -a_1, ..., -a_m\}$ has zeroes (since $f(a_j) = c \Rightarrow a_j \not\in \frac{1}{2}\Lambda, a_j = -a_j$)

Similarly list the poles of $g$ as $\{b_1, ..., b_n, -b_1, ..., -b_n\}$.

Let

$$h(z) = \frac{(\wp(z) - \wp(a_1))(\wp(z) - \wp(a_2))...(\wp(z) - \wp(a_m))}{(\wp(z) - \wp(b_1))...(\wp(z) - \wp(b_n))}$$

Then $h$ is elliptic with the same zeroes and poles as $g$, all of which are simple.

So $\frac{g}{h}$ is an elliptic function without poles so is constant $A$.

$$\frac{g}{h} = A \Rightarrow Ah = \frac{f(z) - c}{f(z) - d}$$
Algebraic Rearrangement
\[ f = Q_1(\wp) \]

If \( f \) is odd then \( f/\wp'(z) \) is an elliptic function s.t. \( f/\wp'(z) = Q_2(z) \) and hence
\[ f = \wp' \cdot Q_2 \]
In general we can write
\[
f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}
\]
and apply the previous cases.

Remark. The upshot of the previous results is that the field of meromorphic functions on \( \mathbb{C}/\Lambda \) is just the field of fractions
\[
\mathbb{C}[x, y]/(y^2 = 4x^3 - g_2x - g_3)
\]
6 The Uniformization Theorem

Definition. A group of homeomorphism of a topological group space \( X \) is said to act properly discontinuously if every \( x \in X \) has an open neighbourhood \( U \) s.t. all \( g(U) \) for \( g \in G \) are pairwise disjoint.

Consider the quotient map \( \pi : X \to X/G \). Continuous wrt the quotient topology.

Given \( x \in X \) and a neighbourhood \( U \) as above \( \pi^{-1}(\pi U) = \coprod g \in G gU \Rightarrow \pi U \) is a bijection and hence a homeomorphism. Thus if \( X \) is path-connected, the \( \pi \) is a regular covering.

Remark. If \( X \) is a Riemann Surface and \( G \subset Aut(X) \) is a subgroup of analytic automorphisms automorphisms acting discontinuously, then \( X/G \) has the structure of a Riemann Surface: we may assume that \((U, \phi)\) is a chart with \( U \) as above then \( \phi \circ (\pi|_{U})^{-1} \) is the region contained on \( X/G \).

This is a conformal structure: Transition functions are analytic since \( G \subset Aut(X) \).

Example. \( \Lambda \subset \mathbb{C} \) is a lattice by translations \( \Rightarrow \mathbb{C}/\Lambda \) is a Riemann Surface.

Note: If \( \tilde{X} \to X \) is a regular covering, the group \( G \) of "deck transformations" acts properly and discontinuously and \( X = \tilde{X}/G \).

6.1 Uniformization Theorem

1. (Hard) Any simply connected Riemann Surface is conformally equivalent to one of \( \mathbb{C}_{\infty}, \mathbb{C} \) and \( \Delta \).

Note that \( \mathbb{C} \) homeomorphic to \( \Delta \) but not conformally equivalent by Liouville.

2. Easier Any Riemann Surface \( R \) is conformally equivalent to \( \tilde{R}/G \) for \( \tilde{R} \) to one of the Riemann Surface in (1) and \( G \subset Aut(\tilde{R}) \), a group of analytic automorphism acting properly and discontinuously. We say that \( R \) is uniformized by \( \tilde{R} \).

Example. \( S^{2} \) has only one conformal structure; if \( R \) is homeomorphic to \( S^{2} \) it is simply connected and hence by (1) is conformally equivalent to \( \mathbb{C}_{\infty} \) Now consider the 3 possibilities for \( \tilde{R} \) in turn

(i) \( \tilde{R} = \mathbb{C}_{\infty} \) \( Aut(\mathbb{C}_{\infty}) \) consists of Möbius transformations \( z \mapsto \frac{az+b}{cz+d} \ ad-bc \neq 0 \) and they have 1 or 2 fixed points \( \Rightarrow \) if \( G \) acts properly discontinuously \( \Rightarrow G = id \Rightarrow R = \mathbb{C}_{\infty} \)

(ii) \( \tilde{R} = \mathbb{C} \), \( Aut(\mathbb{C}) \) consists of maps of the form \( az+b, \ a \in \mathbb{C}^{*} \). If \( a \neq 1 \), \( \exists \) a fixed point so \( G \) consists of translations and so \( G \to \mathbb{C} \) \( g \to g(0) \). Moreover \( G \) consists only of isolated points(Check!). Example Sheet 3 Q1\( \Rightarrow G = \{id\} \). \( \mathbb{C}w_{1} \) on \( \lambda \) lattice. No \( R \) is conformally equivalent to \( \mathbb{C}, \mathbb{C}^{*} \) on \( \mathbb{C}/\Lambda \)(complex torus)
(iii) \( \hat{R} = \Delta \), \( R \) is not conformally equivalent to \( R, \mathbb{C}, \mathbb{C}^* \). Indeed suppose it is. Get \( f \) and using the Monodromy theorem and Liouville, \( f \) is constant. Hence any Riemann Surface which is not \( \mathbb{C}_\infty, \mathbb{C}, \mathbb{C}^* \) or \( \mathbb{C}/\Lambda \) is uniformized by \( \Delta \)

Recall: \( \text{Aut}(\Delta) \) consists of Möbius transformations \( z \mapsto e^{i\theta} \frac{z - a}{1 - \bar{a}z} \) where \( a \in \Delta \). Large literature concerning \( G \subset \text{Aut}(\Delta) \) acting properly discontinuously.

Fuschian Groups

Consequences:

(a) Compact Riemann Surfaces of genus 1 are uniformized by \( \Delta \).

(b) Riemann Mapping Theorem: If \( U \not\subset \mathbb{C} \) domain and simply connected, then it is conformally equivalent to \( \Delta \). Just need to show \( \not\exists \) conformal equivalence \( f : \mathbb{C} \to U \). This is the standard argument saying that \( f \) can’t have an essential singularity. If it did the Casorati-Weierstrass contradicts the injectivity of \( f \).

Hence \( f \) extends to an analytic map \( f : \mathbb{C}_\infty \to f(\mathbb{C}_\infty) \subset \mathbb{C}_\infty \). This is however not surjective.

(c) If a Riemann Surface \( R \) is uniformized by \( \Delta \), \( \exists \) a hyperbolic metric on \( R \) since \( \text{Aut}(\Delta) \) consists of isometries of the hyperbolic metric on \( \Delta \).

Hence if \( R \) is compact, Guass-Bonnet \( \Rightarrow \chi(\hat{R}) < 0 \) so a compact surface of genus 1 must be uniformized by \( \mathbb{C} \).

\( \Rightarrow \) It is a complex torus \( \mathbb{C}/\Lambda \! \).

(d) Picard Theorem If \( f : \mathbb{C} \to \mathbb{C} \setminus \{0, 1\} \) analytic, \( f \) is constant.

\textit{Proof.} Example Sheet 3 Q9 \( \Rightarrow \mathbb{C} \setminus \{0, 1\} \) is uniformized by \( \Delta \). Monodromy Argument \( f \) lifts up to \( \tilde{f} \) (cf. Example Sheet 2 Q3)

Hence by Liouville, \( \tilde{f} \) is constant. \( \Rightarrow f \) is constant. \( \square \)

Any abstract Riemann Surface can be realised as an algebraic object somewhere.