A CLASS OF CLIQUE COVERING MINIMAL GRAPHS

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Throughout the text, $G = (V, E)$ will range over graphs, $\mathcal{D}$ will range over regular ultrafilters over $I$. We say that a sequence $(F_t)_{t \in I}$ covers a set $B$ if $B \subseteq \prod_{t \in I} F_t / \mathcal{D}$. We say that an element $b$ threads $(F_t)_{t \in I}$ if $b \in \prod_{t \in I} F_t / \mathcal{D}$. Recall that whenever $B \subseteq G^l / \mathcal{D}$ is small ($|B| \leq |I|$), there exists a sequence $(H_t)_{t \in I}$ of finite sets covering $B$, by regularity of $\mathcal{D}$.

We say that $\mathcal{D}$ covers the cliques of $G$ if whenever $K \subseteq G^l / \mathcal{D}$ is a small clique, there are is a sequence $(F_t)_{t \in I}$ of cliques covering $K$. We wish to characterize graphs $G$ such that every regular ultrafilter $\mathcal{D}$ covers their cliques. We call such $G$ clique covering minimal or cc-minimal.

If $G$ is cc-minimal, then the graph begotten by blowing up any of $G$’s vertices into a clique of arbitrary size is also cc-minimal. Fix $G$ and let $\{K_v\}_{v \in V}$ be disjoint sets. Let $G' = (V', E')$ be the graph with

$$V' = \bigcup_{v \in V} C_v$$

$$E' = \bigcup_{v \in V} [C_v]^2 \cup \{\{x, y\} \mid x \in C_v, y \in C_u, \{v, u\} \in E\}.$$

**Lemma 1.** A regular ultrafilter $\mathcal{D}$ covers the cliques of $G$ iff it covers the cliques of $G'$.

**Proof.** $\iff$: Choose arbitrarily $x_v \in C_v$ for each $v \in V$. Let $K \subseteq G^l / \mathcal{D}$ be a clique. For each $a \in K$, let $a' = (x_{a[t]})(t \in I)$. Then $K' = \{a' / \mathcal{D} \mid a \in K\}$ is a clique in $G^l / \mathcal{D}$. Let $(F_t)_{t \in I}$ be a clique covering of $K'$ such that $F'_t \subseteq \{x_v \mid v \in V\}$. Then $F_t = \{v \mid x_v \in F'_t\}$ is a clique covering of $K$.

$\implies$: Let $K' \subseteq G^l / \mathcal{D}$ be a clique. For each $a' \in K'$ let $a = (a[t])(t \in I)$ where $a'[t] \in C_{a[t]}$ for each $t \in I$. Let $K = \{a' / \mathcal{D} \mid a' \in K'\}$. Since $K$ is a clique in $G^l / \mathcal{D}$, let $(H_t)_{t \in I}$ be a cover of $K$ by cliques. Let $H'_t = \bigcup_{v \in H_t} C_v$, this is a clique. As $K'$ is small, let $(F'_t)_{t \in I}$ cover $K'$ where $F'_t$ is finite. Then $(H'_t \cap F'_t)_{t \in I}$ is a sequence of finite cliques in $G'$, covering $K'$.

**Corollary 2.** $G$ is cc-minimal iff $G'$ is cc-minimal.

Fix some linear order $\langle P, \leq \rangle$. For each $p \in P$ let $v^1_p$ and $v^0_p$ be two elements. We construct a graph $G(P) = (V, E)$ where

$$V = \{v^1_p \mid p \in P, i = 1, 2\}$$

$$E = \{(v^1_p, v^0_p) \mid p \in P\} \cup \{(v^1_p, v^1_q) \mid q \leq p, i = 1, 2\}$$

For any vertex $v^i_p \in V$ write $r(v^i_p) = p$.

**Proposition 3.** $G(P)$ is cc-minimal

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Proof. Let $\mathcal{D}$ be some regular ultrafilter on some set $I$. Let $N = G(P)^I/\mathcal{D}$ and let $A \subseteq N$ be a clique with $|A| \leq |I|$. For a finite set $F \subseteq G(P)$ denote $r(F) = \min r[F]$.

Claim. There exist finite sets $F_t \subseteq G(P)$ and a point $b \in N$ such that

1. $A \cup \{b\}$ is a clique covered by $(F_t)_{t \in I}$;
2. $b$ threads $(\{x \in F_t \mid r(x) = r(F_t)\})_{t \in I}$.

Proof. Let $(H_t)_{t \in I}$ be a sequence of finite sets covering $A$. We may assume there is no $b \in A$ which threads $(\{x \in H_t \mid r(x) = r(H_t)\})_{t \in I}$.

For each $t \in I$ choose arbitrarily some $h_t \in H_t$ with $r(h_t) = R(H_t)$. Let $F_t = \{h_t\} \cup (H_t \cap N(h_t))$, the neighbourhood of $h_t$ in $H_t$, inclusive of $h_t$.

If $(F_t)_{t \in I}$ covers $A$, then letting $b = \prod_{t \in I} h_t/\mathcal{D}$ we are done. Then assume this is not the case and let $b \in A \setminus (\prod_{t \in I} F_t/\mathcal{D})$. By assumption

$$\{t \in I \mid r(b[t]) > r(h_t)\} \in \mathcal{D}$$

and since $b[t] \notin N(h_t)$ for almost all $t \in I$ it must be that

$$\{t \in I \mid b(t) = v^0_p\} \in \mathcal{D}.$$

Thus, for any $c \in A$, since $\{b, c\}$ is an edge in $N$, we must have

$$\{t \in I \mid r(c[t]) > r(b[t])\} \in \mathcal{D}.$$

Taking $F'_t = \{x \in H_t \mid r(x) \geq r(b[t])\}$, we are done. \qed

Fix $(F_t)_{t \in I}$ and $b$ as in the lemma. Without loss of generality we may assume $F_t \subseteq N_G(b[t])$ and $r(b[t]) = r(F_t)$ for all $t \in I$. Then $F_t$ is a finite clique for all $t \in I$. \qed

Remark. If there exists a minimum $m \in P$, then one can replace the vertices $\{v^1_m, v^0_m\}$ with any cc-minimal graph $G_m$. 

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