Geometry of the pure $n$-ary ab initio Hrushovski construction

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Set Theory, Model Theory and Applications, Eilat 2018
26.04.2018
Definition

A *combinatorial pregeometry* is a (assume countable) set $X$ with a dimension function $d : \mathcal{P}(X) \to \mathbb{N} \cup \{\infty\}$ such that for all $Y, Z \in \mathcal{P}(X), x \in X$:

1. $d(Y) \leq |Y|
2. d(Y) \leq d(Yx) \leq d(Y) + 1$
3. $d(Y \cup Z) \leq d(Y) + d(Z) - d(Y \cap Z)$ (submodular)
4. $d(Y) = \sup \{d(Y_0) \mid Y_0 \in \text{Fin}(Y)\}$ (finitary)

Examples: cardinality, linear dimension, transcendence degree.

Definition

A set $Y$ is *closed* if for any $x$

$$d(Yx) = d(Y) \implies x \in Y.$$
Definition

A finite set \( Y \) is *independent* if \( d(Y) = |Y| \).
An infinite set \( Y \) is independent if all its finite subsets are independent.

A pregeometry is uniquely determined by its set of dependent/independent finite tuples.
A pregeometry can be identified with a first-order structure in the language \( \{ D_k \mid k \in \omega \} \) by interpreting \( D_k \) as the set of dependent \( k \)-tuples.

Definition

A pregeometry on a set \( X \) is *\( n \)-pure* if \( D_n = \emptyset \). Equivalently, every subset of \( X \) of size \( n \) is independent.
A finite set $Y$ is independent if $d(Y) = |Y|$. An infinite set $Y$ is independent if all its finite subsets are independent.

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Fraïssé’s Theorem

Let $\mathbb{C}$ be a countable (up to isomorphisms) class of finitely generated structures, closed under isomorphisms. Let $\leq$ be a distinguished notion of embedding, preserved under isomorphism. Assume

**HP** \quad A \leq B, \ B \in \mathbb{C} \implies A \in \mathbb{C}.

**JEP** \quad A, B \in \mathbb{C} \implies \exists D \in \mathbb{C} \text{ s.t. } A, B \leq D.

**AP** \quad \forall A, B_1, B_2 \in \mathbb{C}

\[ A \leq B_1 \leq B_2 \leq D \]

Then there exists a unique (up to isomorphism) countable generic structure $\mathcal{M}$ with $\text{age}_{\leq}(\mathcal{M}) = \mathbb{C}$ such that $\forall A, B \in \mathbb{C}$

\[ A \leq B \leq \mathcal{M} \]
Definition

For any finite hypergraph $A = (V, E)$ define

$$\delta(A) = |V| - |E|$$

and for any induced subgraph $B \subseteq A$ define

$$d_A(B) = \min \{ \delta(B') \mid B \subseteq B' \subseteq_{\text{fin}} A \}$$

$$B \leq A \iff \delta(B) = d_A(B)$$

The function $d_A$ is the dimension function of a pregeometry on $A$ whenever $\emptyset \leq A$.

Definition

Say that $B$ strongly embeds into $A$ if there exists an embedding $f : B \to A$ such that $f[B] \leq A$. 
Hrushovski’s construction

Let $\mathcal{C} = \{A \mid \emptyset \subseteq A\}$. The class $\mathcal{C}$ is a Fraïssé amalgamation class with respect to $\subseteq$-embeddings. We denote its generic structure $\mathcal{M}$ and its associated pregeometry $G$.

Variations

- Restrict $\mathcal{C}$ to $n$-uniform hypergraphs — $\mathcal{M}^n$
  
  Fact: $n \neq k \implies G^n \not\cong G^k$ (Evans and Ferreira)

- Consider directed hypergraphs — $\mathcal{M}^\neq$
  
  Fact: $G^\neq \cong G$

- Restrict to hypergraphs whose geometries are $n$-pure — $\mathcal{M}_n$

For a fixed $n$, we will construct the pregeometry $\mathcal{M}_n^{n+1}$ — the $n$-pure $(n + 1)$-ary construction — as a Fraïssé-Hrushovski limit.
Hrushovski’s construction

Let \( C = \{ A \mid \emptyset \leq A \} \). The class \( C \) is a Fraïssé amalgamation class with respect to \( \leq \)-embeddings. We denote its generic structure \( M \) and its associated pregeometry \( G \).

Variations

- Restrict \( C \) to \( n \)-uniform hypergraphs — \( M^n \)
  
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- Consider directed hypergraphs — \( M^{\nabla} \)
  
  Fact: \( G^{\nabla} \cong G \)

- Restrict to hypergraphs whose geometries are \( n \)-pure — \( M_n \)

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- Restrict to hypergraphs whose geometries are $n$-pure — $\mathbb{M}_n$

For a fixed $n$, we will construct the pregeometry $\mathbb{M}^{n+1}_n$ — the $n$-pure $(n + 1)$-ary construction — as a Fraïssé-Hrushovski limit.
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- Consider directed hypergraphs — $\mathbb{M}^{\prec}$
  Fact: $\mathbb{G}^{\prec} \cong \mathbb{G}$
- Restrict to hypergraphs whose geometries are $n$-pure — $\mathbb{M}_n$

For a fixed $n$, we will construct the pregeometry $\mathbb{M}_n^{n+1}$ — the $n$-pure $(n + 1)$-ary construction — as a Fraïssé-Hrushovski limit.
Our first goal is assigning $n$-pure pregeometries associated to $(n+1)$-ary hypergraphs a predimension.

**Predimension**

The standard $\delta$ does not work – a graph of a pregeometry may have too many edges. The solution is to calculate predimension using cliques, instead of individual edges.

A closed set $Y$ of dimension $n$ has on it all possible $(n+1)$-edges. Since its dimension should be $n$, we define

$$\lambda(Y) = |Y| - (|Y| - n)$$

Generalizing, for an $n$-pure pregeometry $A$ with $M(A)$ its set of closed sets of dimension $(n+1)$ we define

$$\lambda(A) = |A| - \sum_{C \in M(A)} (|C| - n)$$
Free amalgamation does not work in the case of pregeometries.

**Example**

Let $A$, $B$, $C$ be $(n+1)$-uniform cliques (sets of dimension $n$) with $A \subset B$, $C$, $|A| > n$.
The hypergraph on $B \cup C$ with set of hyperedges $E^B \cup E^C$ is not a hypergraph representing a pregeometry.

The problem is that two distinct “closed sets” have too big of an intersection, and so they must be the same closed set.
Treating a union of intersecting closed sets as a single closed set solves the problem.
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Treating a union of intersecting closed sets as a single closed set solves the problem.
Definition

Define $C_{0}^{geo}$ to be the class of finite $n$-pure pregeometries associated to $(n+1)$-uniform hypergraphs. Define $C^{geo} = \{A \upharpoonright D_{n+1} | A \in C_{0}^{geo}\}$.

With respect to the predimension function $\lambda$ and the altered amalgam, $C^{geo}$ is a Fraïssé-Hrushovski amalgamation class. Denote its generic $M^{geo}$.

Proposition

The pregeometry on $M^{geo}$ given by the predimension function $\lambda$, is precisely $G_{n}^{n+1}$ – the pregeometry given by the predimension function $\delta$ on $M^{n+1}$.

Moreover, the hyperedges of $M^{geo}$ are exactly $D_{n+1}$ of $G_{n}^{n+1}$.

The dependent $(n+1)$-tuples are sufficient information to calculate the dimension of any subset of $G_{n}^{n+1}$, so in some sense we could say that $G_{n}^{n+1}$ is a Fraïssé-Hrushovski limit. However, that is not exactly the case...
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Define $\mathcal{C}_{\text{geo}}^0$ to be the class of finite $n$-pure pregeometries associated to $(n+1)$-uniform hypergraphs. Define $\mathcal{C}_{\text{geo}} = \{ A \upharpoonright D_{n+1} \mid A \in \mathcal{C}_{\text{geo}}^0 \}$.

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The pregeometry on $\mathfrak{M}_{\text{geo}}$ given by the predimension function $\lambda$, is precisely $G_n^{n+1}$ - the pregeometry given by the predimension function $\delta$ on $\mathfrak{M}_n^{n+1}$.

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The pregeometry on $\mathbb{M}^{geo}$ given by the predimension function $\lambda$, is precisely $\mathbb{G}^{n+1}_n$ — the pregeometry given by the predimension function $\delta$ on $\mathbb{M}^{n+1}_n$.
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The structures differ in language: $\mathcal{M}^{\text{geo}}$ is an $n + 1$-uniform hypergraph, whereas $\mathcal{G}_n^{n+1}$ is an expansion that has edges of all arities. Combinatorially, the two hypergraphs hold the exact same information, but this information is not first-order.

Example

$\mathcal{M}^{\text{geo}}$ is $\omega$-stable and saturated. $\mathcal{G}_n^{n+1}$ is not saturated.

Consider the type of a dependent $(n+2)$-tuple whose every $(n+1)$-subtuple is independent.

In particular, the relations $\{D_i\}_{i>n+1}$ are not definable in $\mathcal{M}^{\text{geo}}$. They are strictly $\mathcal{V}$-definable.
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Questions - first order

1. Is $Th(G_n^{n+1})$ $\omega$-stable?
2. Is $Th(G_n^{n+1}) = Th(G_m^{m+1})$ for $m, n \geq 3$?
3. Does $G_n^{n+1}$ have any proper (geometrically) non-trivial reducts?

When the purity and arity differ, say $G_n^{n+2}$, naive generalizations of $\lambda$ fail submodularity. The magic in the case of $(n, n+1)$ is that the class of $n$-pure $(n+1)$-ary pregeometries is closed under substructures.

The construction of $G_n^{n+2}$ can be carried out from a class of pregeometries (Evans-Ferreira) that is not closed under substructures.

Question - predimension construction

Can the construction of $G_n^{n+2}$ be carried out as a Fraïssé-Hrushovski predimension construction?