Geometry and Groups.

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In the 19th century the mathematician Felix Klein formulated what is called the Erlangen program.[1] The idea is that to every kind of geometry there is a group of transformations which preserve the objects of interest in that kind of geometry. These notes give an idea of what he had in mind.

1. Vectors. Two points $p, q \in \mathbb{R}^n$ determine a vector $\mathbf{v} = q - p$. The **length** of that vector is the distance from p to q, i.e.

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \qquad \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n.$$

The **dot product** of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

for $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. Two vectors are said to be **orthogonal** iff their dot product is zero. The **cross product** of two vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is defined by

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1) \in \mathbb{R}^3.$$

2. Lemma. *Any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ satisfy the identities*

$$(\mathbf{u} \cdot \mathbf{v})^2 + |\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2 |\mathbf{v}|^2, \qquad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0.$$

Hence there is a number θ and a vector $\mathbf{n} \in \mathbb{R}^3$ satisfying the equations

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta, & \mathbf{u} \times \mathbf{v} &= (|\mathbf{u}|\,|\mathbf{v}|\,\sin\theta)\,\mathbf{n}, \\ \mathbf{u} \cdot \mathbf{n} &= \mathbf{v} \cdot \mathbf{n} = 0, & |\mathbf{n}| &= 1. \end{aligned}$$

If $\det(\mathbf{u}, \mathbf{v}, \mathbf{n}) > 0$*, the number θ is called the* **angle** *from $\mathbf{u}$ to $\mathbf{v}$ and is unique up to an integer multiple of 2π.*

Proof: Instructive exercise. Use the identity $4ab + (a - b)^2 = (a + b)^2$. □

[1] Click if online.

3. Maps. A **map** f from a set X to a set Y assigns to every point $x \in X$ a point $y = f(x) \in Y$. This is indicated by the notation $f : X \to Y$. The word *function* is often used when $Y = \mathbb{R}$ and the phrase *vector-valued function* is often used when $Y = \mathbb{R}^n$. A map $f : X \to Y$ is **invertible** iff it has a two sided inverse, i.e. iff there is a map $f^{-1} : Y \to X$ such that

$$y = f(x) \iff x = f^{-1}(y)$$

for $x \in X$ and $y \in Y$.

4. Transformations. An **affine transformation** is a map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ of form

$$p = \Phi(q) = Uq + \mathbf{c}$$

for $p, q \in \mathbb{R}^n = \mathbb{R}^{n\times 1}$ where $U \in \mathbb{R}^{n\times n}$ is an invertible matrix and $\mathbf{c} \in \mathbb{R}^{n\times 1}$. An affine transformation is called

- a **translation** iff U is the identity matrix, i.e. $\Phi(q) = q + \mathbf{c}$.
- a **linear transformation** iff $\mathbf{c} = 0$.
- **orientation preserving** iff $\det(U) > 0$.
- an **orthogonal transformation** iff $U^* = U^{-1}$ and $\mathbf{c} = 0$.
- an **isometry** iff $U^* = U^{-1}$.
- a **rigid motion** iff $U^* = U^{-1}$ and $\det(U) = 1$.

(Here U^* is the transpose of U.)

5. Groups. A set G of invertible maps $\Phi : X \to X$ from a set X to itself is a **group** iff it satisfies the following three conditions:

(i) The identity map is an element of G.

(ii) The inverse Φ^{-1} of an element of G is an element of G.

(iii) The composition $\Phi_1 \circ \Phi_2$ of two elements of G an element of G.

6. Theorem. *The affine transformations of $\mathbb{R}^n$ form a group.*

Proof: (i) If U is the identity matrix and $\mathbf{c} = 0$ then $\Phi(p) = p$ for $p \in \mathbb{R}^n$.

(ii) $\Phi^{-1}(p) = U^{-1}(p - \mathbf{c}) = U^{-1}p - U^{-1}\mathbf{c}$.

(iii) $\Phi_1 \circ \Phi_2(q) = U_1U_2p + (U_2\mathbf{c}_1 + \mathbf{c}_2)$. □

7. Corollary. *Each of the subsets defined in n°$\underline{4}$ above is a group.*

Proof: Easy exercise. □

8. Theorem. *An orthogonal transformation of* $U : \mathbb{R}^n \to \mathbb{R}^n$ *preserves dot products and (therefore) lengths, i.e.*

$$(U\mathbf{u}) \cdot (U\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}, \qquad |U\mathbf{v}| = |\mathbf{v}|$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *If* $n = 3$ *an orientation preserving orthogonal transformation* U *preserves the cross product, i.e.*

$$(U\mathbf{u}) \times (U\mathbf{v}) = U(\mathbf{u} \times \mathbf{v})$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

Proof: Assume that $U^* = U^{-1}$. The first statement follows from the formula

$$(U\mathbf{u}) \cdot (U\mathbf{v}) = (U^* U\mathbf{u}) \cdot \mathbf{v}.$$

Since $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ and $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$ it follows that U preserves lengths. From Lemma n°2 we get $(U\mathbf{u}) \times (U\mathbf{v}) = \pm U(\mathbf{u} \times \mathbf{v})$. Now the identities

$$U\big(\mathbf{u}, \mathbf{v}, \mathbf{w}\big) = \big(U\mathbf{u}, U\mathbf{v}, U\mathbf{w}\big), \qquad \det(UV) = \det(U)\det(V),$$

and

$$\det(\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}) = |\mathbf{u} \times \mathbf{v}|^2$$

imply that $(U\mathbf{u}) \times (U\mathbf{v}) = U(\mathbf{u} \times \mathbf{v})$ when $\det(U) > 0$. □

9. Theorem. *A map* $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ *is*

(i) *an affine transformation if and only if it preserves parameterized lines, i.e.*

$$\Phi((1-t)p + tq) = (1-t)\Phi(p) + t\Phi(q)$$

(ii) *a translation if and only if its preserves vectors, i.e.*

$$\Phi(p) - \Phi(q) = p - q;$$

(iii) *an isometry if and only if it preserves distance, i.e.*

$$|\Phi(p) - \Phi(q)| = |p - q|;$$

(iv) *a rigid motion if and only if it preserves distance and orientation.*

Proof: This follows easily from the following computations.

(i) $U((1-t)p + tq) + \mathbf{c} = (1-t)(Up + \mathbf{c}) + t(Uq + \mathbf{c})$.

(ii) $(p + \mathbf{c}) - (q + \mathbf{c}) = p - q$.

(iii) $|\Phi(p) - \Phi(q)|^2 = (U^*U(p-q)) \cdot (p-q)) = (p-q) \cdot (p-q) = |p-q|^2$.

(iv) $\det(U^*) = \det(U)$ and $\det(U^{-1}) = \det(U)^{-1}$ so if $U^* = U^{-1}$ and $\det(U) > 0$ then $\det(U) = 1$. □

10. Example. When $n = 2$ a rigid motion Φ has the form

$$\begin{array}{rcl} x & = & (\cos\theta)\xi - (\sin\theta)\eta + x_0 \\ y & = & (\sin\theta)\xi + (\cos\theta)\eta + y_0 \end{array}$$

where $q = (\xi, \eta)$, $p = (x, y)$, $p = \Phi(q)$, $\mathbf{c} = (x_0, y_0)$.

11. Theorem. *Let Γ be a plane curve defined by quadratic equation, say*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 : Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\}.$$

Assume that $B^2 - 4AC < 0$ and that Γ contains at least two distinct points. Then there is a rigid motion Φ which transforms Γ to an ellipse in standard form, i.e.

$$\Phi^{-1}(\Gamma) = \left\{(\xi, \eta) \in \mathbb{R}^2 : \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1\right\}.$$

Moreover $B^2 - 4AC = B_1^2 - 4A_1C_1$ and $A + C = A_1 + C_1$ where $B_1 = 0$, $A_1 = a^{-2}$, $C_1 = b^{-2}$.

Proof: Plug in and solve for θ to make the $\xi\eta$ term go away. Then complete the squares ξ^2 and η^2 and solve for x_0 and y_0 to make the ξ and η terms go away. For more details see any textbook with a title like *Calculus and Analytic Geometry.* □