Partitions of Unity

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1. Definition. Let X be a topological space and let I be a (possibly uncountable) index set. A collection $\{X_i\}_{i\in I}$ of subsets of X is called **locally finite** iff each $x \in X$ has a neighborhood U such that $U \cap X_i \neq \emptyset$ for at most finitely many $i \in I$.

2. Remark. If the collection $\{X_i\}_{i\in I}$ is locally finite then so is the collection $\{\overline{X_i}\}_{i\in I}$. Moreover

$$\bigcup_{i\in I} \overline{X_i} = \overline{\bigcup_{i\in I} X_i}.$$

3. Definition. The **support** of a function $f : X \to \mathbb{R}$ is the set $\operatorname{supp}(f)$ defined by

$$\operatorname{supp}(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

Thus a point is not in $\operatorname{supp}(f)$ iff it has a neighborhood on which f vanishes identically. A collection $\{f_i\}_{i\in I}$ of real valued functions on X is called **locally finite** iff the collection $\{\operatorname{supp}(f_i)\}_{i\in I}$ of supports is locally finite.

4. Definition. The space X is called **sigma-compact** iff it is a countable union of compact subsets.

5. Lemma. If a topological space X is locally compact and sigma compact, then $X = \bigcup_{i=1}^{\infty} K_i$ where each K_i is compact and $K_i \subset \operatorname{int} K_{i+1}$ for $i \in \mathbb{N}$.

6. Definition. A **partition of unity** on X is a collection $\{g_i\}_i$ of continuous real valued functions on X such that

(1) $g_i \geq 0$ for each i;

(2) every $x \in X$ has a neighborhood U such that $U \cap \operatorname{supp}(g_i) = \varphi$ for all but finitely many of the g_i;

(3) for each $x \in X$

$$\sum_i g_i(x) = 1.$$

Note that by (2), the sum in (3) is finite.

7. Definition. A partition of unity $\{g_i\}_i$ on X is **subordinate** to an open cover of X iff for each g_i there is an element U of the cover such that $\mathrm{supp}(g_i) \subset U$. The space X **admits** partitions of unity iff for every open cover of X there is a partition of unity subordinate to the cover.

8. Remark. A Hausdorff space admits continuous partitions of unity if and only if it is paracompact, i.e. iff every cover has a locally finite refinement.

9. Definition. A C^r manifold is a set X equipped with a maximal atlas of charts. Each chart is a bijection $\varphi : U \to \varphi(U)$ where $U \subseteq M$ is a set and $\varphi(U) \subseteq \mathbb{R}^m$ is an open set. Any two charts (φ, U) and (ψ, V) in the atlas a compatible in the sense that the sets $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^m$ and the overlap map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is a C^r diffeomorphism. The atlas is maximal in the sense that every chart which is compatible with all the charts in the atlas is already contained in the atlas. The domains of the charts generate a topology on X and it is always assumes that this topology is Hausdorff and second countable.

10. Theorem. A C^r manifold admits partitions of unity.

11. Remark. The proof of the theorem shows that a C^r manifold admits C^r partitions of unity (not just continuous ones) in the sense that for every open cover of X there is a C^r partition of unity subordinate to the cover.

12. Corollary. A (second countable Hausdorff) manifold is paracompact.

13. Corollary. (C^r Urysohn's Lemma) Let A and B be disjoint closed subsets of a C^r manifold X. Then there is a C^r function $g : X \to \mathbb{R}$ such that $0 \leq g(x) \leq 1$ for all $x \in A$, $g(x) = 0$ for $x \in X$ and $g(x) = 1$ for $x \in B$.

14. Corollary. Any closed subset of a C^r manifold is the zero set of a C^r real-valued function.

15. Definition. A topological space X has **covering dimension at most** m iff every open cover of X has a refinement with the property that each point of X is contained in at most $m + 1$ members of the refinement. The **covering dimension** of X is the least m such that X has covering dimension at most m; it is ∞ if there is no such m.

16. Theorem. An m-dimensional manifold has covering dimension m.

17. Proposition. Let X be a paracompact space with covering dimension m. Then every open cover of X has a refinement which is the union of collections $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m$ such that any of the elements of each $\mathcal{C}_i$ ($i = 0, 1, \ldots, m$) are pairwise disjoint.

18. Corollary. An m-dimensional manifold can be covered by $m + 1$ charts.

19. Theorem Let X be an m-dimensional C^r manifold ($r \geq 0$) and let $k = (m+1)^2$. Then there exists a C^r closed embedding $f : X \to R^k$.

20. Corollary. A manifold is metrizable.

21. Theorem. The following conditions on a connected topological manifold X are equivalent:

(1) X is metrizable.

(2) X is paracompact.

(3) X has a countable basis.

(4) X admits partitions of unity.

(5) X is sigma-compact.

(6) X has finite covering dimension.

(7) X admits a proper embedding into some Euclidean space.