Lecture 13 : UI MGs: Optional Sampling Thm

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References: [Wil91, Appendix to Chapter 14], [Dur10, Section 4.7].

1 Review: Stopping times

Recall:

DEF 13.1 A random variable $T : \Omega \to \mathbb{Z}_+ \equiv \{0, 1, \ldots, +\infty\}$ is called a stopping time if

$$\{T = n\} \in \mathcal{F}_n, \forall n \in \mathbb{Z}_+.$$

EX 13.2 Let $\{A_n\}$ be an adapted process and $B \in \mathcal{B}$. Then

$$T = \inf\{n \geq 0 : A_n \in B\},$$

is a stopping time.

THM 13.3 (Optional Stopping Thm) Let $\{M_n\}$ be a MG and $T$ be a stopping time. Then $M_T$ is integrable and

$$\mathbb{E}[M_T] = \mathbb{E}[X_0].$$

if one of the following holds:

1. $T$ is bounded.
2. $M$ is bounded and $T$ is a.s. finite.
3. $\mathbb{E}[T] < +\infty$ and $M$ has bounded increments.
4. $M$ is UI.
The σ-field $\mathcal{F}_T$

DEF 13.4 ($\mathcal{F}_T$) Let $T$ be a stopping time. Denote by $\mathcal{F}_T$ the set of all events $F$ such that $\forall n \in \mathbb{Z}_+$

$$F \cap \{T = n\} \in \mathcal{F}_n.$$ 

The following two lemmas clarify the definition:

LEM 13.5 $\mathcal{F}_T = \mathcal{F}_n$ if $T \equiv n$, $\mathcal{F}_T = \mathcal{F}_\infty$ if $T \equiv \infty$ and $\mathcal{F}_T \subseteq \mathcal{F}_\infty$ for any $T$.

Proof: In the first case, note $F \cap \{T = k\}$ is empty if $k \neq n$ and is $F$ if $k = n$. So if $F \in \mathcal{F}_T$ then $F = F \cap \{T = n\} \in \mathcal{F}_n$ and if $F \in \mathcal{F}_n$ then $F = F \cap \{T = n\} \in \mathcal{F}_n$. Moreover $\emptyset \in \mathcal{F}_n$ so we have proved both inclusions. This works also for $n = \infty$. For the third claim note

$$F = \bigcup_{k \in \mathbb{Z}_+} F \cap \{T = n\} \in \mathcal{F}_\infty.$$ 


LEM 13.6 If $X$ is adapted and $T$ is a stopping time then $X_T \in \mathcal{F}_T$ (where we assume that $X_\infty \in \mathcal{F}_\infty$, e.g., $X_\infty = \liminf X_n$).

Proof: For $B \in \mathcal{B}$

$$\{X_T \in B\} \cap \{T = n\} = \{X_n \in B\} \cap \{T = n\} \in \mathcal{F}_n.$$ 


LEM 13.7 If $S, T$ are stopping times then $\mathcal{F}_{S \land T} \subseteq \mathcal{F}_T$.

Proof: Let $F \in \mathcal{F}_{S \land T}$. Note that

$$F \cap \{T = n\} = \bigcup_{k \leq n} [(F \cap \{S \land T = k\}) \cap \{T = n\}] \in \mathcal{F}_n.$$ 


3 Optional Sampling Theorem (OST)

THM 13.8 (Optional Sampling Theorem) If $M$ is a UI MG and $S, T$ are stopping times with $S \leq T$ a.s. then $\mathbb{E}|M_T| < +\infty$ and

$$\mathbb{E}[M_T | \mathcal{F}_S] = M_S.$$
Proof: Since $M$ is UI, $\exists M_{\infty} \in L^1$ s.t. $M_n \to M_{\infty}$ a.s. and in $L^1$. We prove a more general claim:

**LEM 13.9**

$$\mathbb{E}[M_{\infty} \mid F_T] = M_T.$$ 

Indeed, we then get the theorem by (TOWER) and (JENSEN).

**Proof:** (Lemma) Wlog we assume $M_{\infty} \geq 0$ so that $M_n = \mathbb{E}[M_{\infty} \mid F_n] \geq 0 \ \forall n$. Let $F \in \mathcal{F}_T$. Then (trivially)

$$\mathbb{E}[M_{\infty}; F \cap \{T = \infty\}] = \mathbb{E}[M_T; F \cap \{T = \infty\}]$$

so STS

$$\mathbb{E}[M_{\infty}; F \cap \{T < +\infty\}] = \mathbb{E}[M_T; F \cap \{T < +\infty\}].$$

In fact, by (MON), STS

$$\mathbb{E}[M_{\infty}; F \cap \{T \leq k\}] = \mathbb{E}[M_T; F \cap \{T \leq k\}] = \mathbb{E}[M_{T \land k}; F \cap \{T \leq k\}],$$

$\forall k$. To conclude we make two observations:

1. **$F \cap \{T \leq k\} \in \mathcal{F}_{T \land k}$.** Indeed if $n \leq k$

   $$F \cap \{T \leq k\} \cap \{T \land k = n\} = F \cap \{T = n\} \in \mathcal{F}_n,$$

   and if $n > k$

   $$= \emptyset \in \mathcal{F}_n.$$

2. **$\mathbb{E}[M_{\infty} \mid \mathcal{F}_{T \land k}] = M_{T \land k}$.** Since $\mathbb{E}[M_{\infty} \mid F_k] = M_k$, STS $\mathbb{E}[M_k \mid \mathcal{F}_{T \land k}] = M_{T \land k}$. But note that if $G \in \mathcal{F}_{T \land k}$

   $$\mathbb{E}[M_k; G] = \sum_{l \leq k} \mathbb{E}[M_k; G \cap \{T \land k = l\}] = \sum_{l \leq k} \mathbb{E}[M_l; G \cap \{T \land k = l\}] = \mathbb{E}[M_{T \land k}; G]$$

   since $G \cap \{T \land k = l\} \in \mathcal{F}_l$.

\[ \blacksquare \]

4 Example: Biased RW

**DEF 13.10** The asymmetric simple RW with parameter $1/2 < p < 1$ is the process $\{S_n\}_{n \geq 0}$ with $S_0 = 0$ and $S_n = \sum_{k \leq n} X_k$ where the $X_k$s are iid in $\{-1, +1\}$ s.t. $\mathbb{P}[X_1 = 1] = p$. Let $q = 1 - p$. Let $\phi(x) = (q/p)x$ and $\psi_n(x) = x - (p - q)n$. 
**THM 13.11** Let \( \{S_n\} \) as above. Let \( a < 0 < b \). Define \( T_x = \inf \{n \geq 0 : S_n = x\} \). Then

1. We have \( \mathbb{P}[T_a < T_b] = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}. \)

   In particular, \( \mathbb{P}[T_a < +\infty] = 1/\phi(a) \) and \( \mathbb{P}[T_b < +\infty] = 1. \)

2. We have \( \mathbb{E}[T_b] = \frac{b}{2p - 1}. \)

**Proof:** There are two MGs here:

\[
\mathbb{E}[\phi(S_n) | F_{n-1}] = p(q/p)^{S_n-1+1} + q(q/p)^{S_n-1-1} = \phi(S_n-1),
\]

and

\[
\mathbb{E}[\psi_n(S_n) | F_{n-1}] = p[S_n-1+1-(p-q)(n)] + q[S_n-1-1-(p-q)(n)] = \psi_{n-1}(S_n-1).
\]

Let \( N = T_a \wedge T_b \). Now note that \( \phi(S_{N \wedge n}) \) is a bounded MG and therefore applying the MG property at time \( n \) and taking limits as \( n \to \infty \) (using (DOM))

\[
\phi(0) = \mathbb{E}[\phi(S_N)] = \mathbb{P}[T_a < T_b] \phi(a) + \mathbb{P}[T_a > T_b] \phi(b),
\]

where we need to prove that \( N < +\infty \) a.s. Indeed, since \( (b-a) +1 \)-steps always take us out of \( (a,b) \),

\[
\mathbb{P}[T_b > n(b-a)] \leq (1 - q^{b-a})^n,
\]

so that

\[
\mathbb{E}[T_b] = \sum_{k \geq 0} \mathbb{P}[T_b > k] \leq \sum_{n} (b-a)(1 - q^{b-a})^n < +\infty.
\]

In particular \( T_b < +\infty \) a.s. and \( N < +\infty \) a.s. Rearranging the formula above gives the first result. (For the second part of the first result, take \( b \to +\infty \) and use monotonicity.)

For the third one, note that \( T_b \wedge n \) is bounded so that

\[
0 = \mathbb{E}[S_{T_b \wedge n} - (p-q)(T_b \wedge n)].
\]

By (MON), \( \mathbb{E}[T_b \wedge n] \uparrow \mathbb{E}[T_b] \). Finally, using

\[
\mathbb{P}[- \inf_n S_n \geq -a] = \mathbb{P}[T_n < +\infty],
\]

and the fact that \( - \inf_n S_n \geq 0 \) shows that \( \mathbb{E}[- \inf_n S_n] < +\infty \). Hence, we can use (DOM) with \( |S_{T_b \wedge n}| \leq \max\{b, -\inf_n S_n\}. \)
References
