1 Invariance

We begin with some useful invariance properties. The following are immediate.

THM 19.1 (Time translation) Let \( s \geq 0 \). If \( B(t) \) is a standard Brownian motion, then so is \( X(t) = B(t + s) - B(s) \).

THM 19.2 (Scaling invariance) Let \( a > 0 \). If \( B(t) \) is a standard Brownian motion, then so is \( X(t) = a^{-1} B(a^2 t) \).

**Proof:** Sketch. We compute the variance of the increments:

\[
\text{Var}[X(t) - X(s)] = \text{Var}[a^{-1}(B(a^2 t) - B(a^2 s))] \\
= a^{-2}(a^2 t - a^2 s) \\
= t - s.
\]

THM 19.3 (Time inversion) If \( B(t) \) is a standard Brownian motion, then so is

\[
X(t) = \begin{cases} 
0, & t = 0, \\
tB(t^{-1}), & t > 0.
\end{cases}
\]

**Proof:** Sketch. We compute the covariance function for \( s < t \):

\[
\text{Cov}[X(s), X(t)] = \text{Cov}[sB(s^{-1}), tB(t^{-1})] \\
= st (s^{-1} \land t^{-1}) \\
= s.
\]
It remains to check continuity at 0. Note that
\[
\left\{ \lim_{t \downarrow 0} B(t) = 0 \right\} = \bigcap_{m \geq 1} \bigcup_{n \geq 1} \left\{ |B(t)| \leq 1/m, \forall t \in \mathbb{Q} \cap (0, 1/n) \right\},
\]
and
\[
\left\{ \lim_{t \downarrow 0} X(t) = 0 \right\} = \bigcap_{m \geq 1} \bigcup_{n \geq 1} \left\{ |X(t)| \leq 1/m, \forall t \in \mathbb{Q} \cap (0, 1/n) \right\}.
\]

The RHSs have the same probability because the distributions on all finite-dimensional sets — and therefore on the rationals — are the same. The LHS of the first one has probability 1.

Typical applications of these are:

**COR 19.4** For \( a < 0 < b \), let
\[
T(a, b) = \inf \{ t \geq 0 : B(t) \in \{ a, b \} \}.
\]

Then
\[
\mathbb{E}[T(a, b)] = a^2 \mathbb{E}[T(1, b/a)].
\]

In particular, \( \mathbb{E}[T(-b, b)] \) is a constant multiple of \( b^2 \).

**Proof:** Let \( X(t) = a^{-1}B(a^2 t) \). Then,
\[
\mathbb{E}[T(a, b)] = a^2 \mathbb{E}[\inf\{ t \geq 0 : X(t) \in \{ 1, b/a \} \}]
= a^2 \mathbb{E}[T(1, b/a)].
\]

**COR 19.5** Almost surely,
\[
t^{-1}B(t) \to 0.
\]

**Proof:** Let \( X(t) \) be the time inversion of \( B(t) \). Then
\[
\lim_{t \to \infty} \frac{B(t)}{t} = \lim_{t \to \infty} X(1/t) = X(0) = 0.
\]
Modulus of continuity

By construction, $B(t)$ is continuous a.s. In fact, we can prove more.

**DEF 19.6 (Hölder continuity)** A function $f$ is said locally $\alpha$-Hölder continuous at $x$ if there exists $\varepsilon > 0$ and $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^{\alpha},$$

for all $y$ with $|y - x| < \varepsilon$. We refer to $\alpha$ as the Hölder exponent and to $c$ as the Hölder constant.

**THM 19.7 (Holder continuity)** If $\alpha < 1/2$, then almost surely Brownian motion is everywhere locally $\alpha$-Hölder continuous.

**Proof:**

**LEM 19.8** There exists a constant $C > 0$ such that, almost surely, for every sufficiently small $h > 0$ and all $0 \leq t \leq 1 - h$,

$$|B(t + h) - B(t)| \leq C \sqrt{h \log(1/h)}.$$

**Proof:** Recall our construction of Brownian motion on $[0, 1]$. Let

$$D_n = \{k2^{-n} : 0 \leq k \leq 2^n\},$$

and

$$D = \bigcup_{n=0}^{\infty} D_n.$$

Note that $D$ is countable and consider $\{Z_t\}_{t \in D}$ a collection of independent standard Gaussians. Let

$$F_0(t) = \begin{cases} 
Z_1, & t = 1, \\
0, & t = 0,
\end{cases}$$

linearly, in between.

and for $n \geq 1$

$$F_n(t) = \begin{cases} 
2^{-(n+1)/2}Z_t, & t \in D_n \setminus D_{n-1}, \\
0, & t \in D_{n-1},
\end{cases}$$

linearly, in between.

Finally

$$B(t) = \sum_{n=0}^{\infty} F_n(t).$$
Each $F_n$ is piecewise linear and its derivative exists almost everywhere. By construction, we have

$$\|F'_n\|_\infty \leq \frac{\|F_n\|_\infty}{2^{-n}}.$$ 

Recall that there is $N$ (random) such that $|Z_d| < c\sqrt{n}$ for all $d \in D_n$ with $n > N$. In particular, for $n > N$ we have

$$\|F_n\|_\infty < c\sqrt{n^{2-(n+1)/2}}.$$ 

Using the mean-value theorem, assuming $l > N$,

$$|B(t+h) - B(t)| \leq \sum_{n=0}^{\infty} |F_n(t+h) - F_n(t)|$$

$$\leq \sum_{n=0}^{l} h\|F'_n\|_\infty + \sum_{n=l+1}^{\infty} 2\|F_n\|_\infty,$$

$$\leq h \sum_{n=0}^{N} \|F'_n\|_\infty + c\sum_{n=N}^{l} \sqrt{n2^{n/2}} + 2c \sum_{n=l+1}^{\infty} \sqrt{n2^{-n/2}}.$$ 

Take $h$ small enough that the first term is smaller than $\sqrt{\log(1/h)}$ and $l$ defined by $2^{-l} < h \leq 2^{-l+1}$ exceeds $N$. Then approximating the second and third terms by their largest element gives the result.

We go back to the proof of the theorem. For each $k$, we can find an $h(k)$ small enough so that the result applies to the standard BMs

$$\{B(k+t) - B(k) : t \in [0, 1]\},$$

and

$$\{B(k+1-t) - B(k+1) : t \in [0, 1]\}.$$ 

Since there are countably many intervals $[k, k+1)$, such $h(k)$’s exist almost surely on all intervals simultaneously. Then note that for any $\alpha < 1/2$, if $t \in [k, k+1)$ and $h < h(k)$ small enough,

$$|B(t+h) - B(t)| \leq C\sqrt{\log(1/h)} \leq Ch^{\alpha}(= Ch^{1/2}(1/h^{1/2-\alpha})).$$

This concludes the proof.

In fact:

**THM 19.9 (Lévy’s modulus of continuity)** Almost surely,

$$\limsup_{h \downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log(1/h)}} = 1.$$ 

For the proof, see [MP10].

This result is tight. See [MP10, Remark 1.21].
3 Non-Monotonicity

**THM 19.10** Almost surely, for all $0 < a < b < +\infty$, standard BM is not monotone on the interval $[a, b]$.

**Proof:** It suffices to look at intervals with rational endpoints because any general non-degenerate interval of monotonicity must contain one of those. Since there are countably many rational intervals, it suffices to prove that any particular one has probability 0 of being monotone. Let $[a, b]$ be such an interval. Note that for any finite sub-division

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b,$$

the probability that each increment satisfies

$$B(a_i) - B(a_{i-1}) \geq 0, \quad \forall i = 1, \ldots, n,$$

or the same with negative, is at most

$$2 \left( \frac{1}{2} \right)^n \to 0,$$

as $n \to \infty$ by symmetry of Gaussians.

More generally, we can prove the following. For a proof see [Lig10].

**THM 19.11** Almost surely, BM satisfies:

1. The set of times at which local maxima occur is dense.
2. Every local maximum is strict.
3. The set of local maxima is countable.

**Proof:** Part (3). We use part (2). If $t$ is a strict local maximum, it must be in the set

$$\bigcup_{n=1}^{+\infty} \{ t : B(t, \omega) > B(s, \omega), \forall s, |s - t| < n^{-1} \}.$$

But for each $n$, the set must be countable because two such $t$’s must be separated by $n^{-1}$. So the union is countable.

**Further reading**

Other constructions in [Dur10, Section8.1] and [Lig10, Section 1.5]. Proof of modulus of continuity [MP10, Theorem 1.14].
References

