1 Random walks and recurrence

DEF 28.1 A random walk (RW) on \( \mathbb{R}^d \) is an SP of the form:

\[
S_n = \sum_{i \leq n} X_i, \quad n \geq 1
\]

where the \( X_i \)s are iid in \( \mathbb{R}^d \).

EX 28.2 (SRW on \( \mathbb{Z}^d \)) This is the special case:

\[
P[X_i = e_j] = P[X_i = -e_j] = \frac{1}{2d},
\]

for all \( j = 1, \ldots, d \) where \( e_j \) is the unit vector in the \( j \)-th direction.

DEF 28.3 We say that \( x \in \mathbb{R}^d \) is a recurrent value if, for all \( \varepsilon > 0 \), \( P[\|S_n - x\| < \varepsilon \text{ i.o.}] = 1 \). Let \( V \) be the set of recurrent values. We say that \( S_n \) is transient if \( V = \emptyset \), o.w. it is recurrent.

2 SRW on \( \mathbb{Z} \)

Recall Stirling’s formula:

\[
n! \sim n^ne^{-n}\sqrt{2\pi n}.
\]

THM 28.4 (SRW on \( \mathbb{Z} \)) SRW on \( \mathbb{Z} \) is recurrent.

Proof: First note the periodicity. So we look at \( S_{2n} \). Then

\[
P[S_{2n} = 0] = \binom{2n}{n} 2^{-2n}
\]

\[
\sim 2^{-2n} \left( \frac{(2n)^{2n}}{(n^n)^2} \right)^2 \frac{\sqrt{2n}}{\sqrt{2\pi n}}
\]

\[
\sim \frac{1}{\sqrt{\pi n}}.
\]
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So
\[ \sum_m \mathbb{P}[S_m = 0] = \infty. \]

Denote
\[ T_0^{(n)} = \inf \{ m > T_0^{(n-1)} : S_m = 0 \}. \]

By the strong Markov property \( \mathbb{P}[T_0^{(n)} < \infty] = \mathbb{P}[T_0 < \infty]^n. \) Note that
\[ \sum_m \mathbb{P}[S_m = 0] = \mathbb{E}[\sum_m 1_{S_m = 0}] = \mathbb{E}[\sum_n 1_{T_0^{(n)} < \infty}] = \sum_n \mathbb{P}[T_0^{(n)} < \infty] = \sum_n \mathbb{P}[T_0 < \infty]^n = \frac{1}{1 - \mathbb{P}[T_0 < \infty]}.
\]

So \( \mathbb{P}[T_0 < \infty] = 1. \)

3 SRW on \( \mathbb{Z}^2 \)

Now \( X_1 \) is in \( \mathbb{Z}^2 \) and \( \mathbb{P}[X_1 = (1, 0)] = \cdots = \mathbb{P}[X_1 = (0, -1)] = 1/4. \)

THM 28.5 (SRW on \( \mathbb{Z}^2 \)) SRW on \( \mathbb{Z}^2 \) is recurrent.

Proof: Let \( R_n = (S_n^{(1)}, S_n^{(2)}) \) where \( S_n^{(i)} \) are independent SRW on \( \mathbb{Z} \). Note that \( R_n \) is a SRW on \( \mathbb{Z}^2 \) rotated by 45 degrees. So the probability to be back at \( (0, 0) \) is the same as for two independent SRW on \( \mathbb{Z} \) to be back at 0 simultaneously. Therefore,
\[ \mathbb{P}[S_{2n} = (0, 0)] = \mathbb{P}[S_{2n}^{(1)} = 0]^2 \sim \frac{1}{\pi n}, \]
whose sum diverges. \( \blacksquare \)

4 SRW on \( \mathbb{Z}^3 \)

Now \( X_1 \) is in \( \mathbb{Z}^3 \) and \( \mathbb{P}[X_1 = (1, 0, 0)] = \cdots = \mathbb{P}[X_1 = (0, 0, -1)] = 1/6. \)

THM 28.6 (SRW on \( \mathbb{Z}^3 \)) SRW on \( \mathbb{Z}^3 \) is transient.
Proof: Note, since the number of steps in opposite directions has to be equal,

\[ \mathbb{P}[S_{2n} = 0] = 6^{-2n} \sum_{j,k} \frac{(2n)!}{j!k!(n-k-j)!}^2 \]

\[ = 2^{-2n} \binom{2n}{n} \sum_{j,k} \frac{n!}{j!k!(n-k-j)!} \]

\[ \leq 2^{-2n} \binom{2n}{n} \max_{j,k} \frac{n!}{j!k!(n-k-j)!}, \]

where we used that \( \sum_{j,k} a_{j,k}^2 \leq \max_{i,j} a_{j,k} \equiv a^* \) if \( \sum_{j,k} a_{j,k} = 1 \) and \( a_{j,k} \geq 0 \).

Note that if \( j < n/3 \) and \( k > n/3 \) then

\[ \frac{(j+1)!(k-1)!}{j!k!} = \frac{j+1}{k} \leq 1. \]

That implies that the term in the max is maximized when \( j, k, (n-k-j) \) are roughly \( n/3 \). Using Stirling

\[ \frac{n!}{j!k!(n-k-j)!} \sim \frac{n^n}{j^j k^k (n-k-j)^{n-k-j}} \sqrt{\frac{n}{j!k!(n-k-j)!}} \frac{1}{2\pi} \sim C \frac{3^n}{n}. \]

Hence \( \mathbb{P}[S_{2n} = 0] \sim C n^{-3/2} \) which is summable and \( \mathbb{P}[T_0 < \infty] < 1. \)

**Corollary 28.7** SRW on \( \mathbb{Z}^d \) with \( d > 3 \) is transient.

**Proof:** Let \( R_n = (S_n^1, S_n^2, S_n^3) \). Let

\[ U_m = \inf\{n > U_{m-1} : R_n \neq R_{U_{m-1}} \}. \]

Then \( R_{U_n} \) is a three-dimensional SRW. It visits \( (0,0,0) \) only finitely many times whp.

**5 RW in \( \mathbb{R}^d \)**

Now \( X_1 \) is in \( \mathbb{R}^d \). See [Dur10, Section 3.2] for a proof of:

- \( S_n \) is recurrent in \( d = 1 \) if \( S_n/n \to 0 \) in probability
- \( S_n \) is recurrent in \( d = 2 \) if \( S_n/\sqrt{n} \Rightarrow \) Gaussian
- \( S_n \) is recurrent in \( d \geq 3 \) if it is truly three-dimensional (for all \( \theta \neq 0 \), \( \mathbb{P}[X_1 \cdot \theta \neq 0] > 0 \))
References