CHARACTERIZATIONS OF COMPACTNESS
FOR METRIC SPACES

Definition. Let $X$ be a metric space with metric $d$.
(a) A collection $\{G_\alpha\}_{\alpha \in A}$ of open sets is called an open cover of $X$ if every $x \in X$ belongs to at least one of the $G_\alpha$, $\alpha \in A$. An open cover is finite if the index set $A$ is finite.
(b) $X$ is compact if every open cover of $X$ contains a finite subcover.

Definition. Let $X$ be a metric space with metric $d$ and let $A \subset X$. We say that $A$ is a compact subset if the metric space $A$ with the inherited metric $d$ is compact.

Examples: Any finite metric space is compact.
As an exercise show directly from the definition that the subset $K$ of $\mathbb{R}$ consisting of 0 and the numbers $1/n$, $n = 1, 2, \ldots$ is compact.

Definition. A subset $A$ of $X$ is relatively compact if the closure $\overline{A} \subset X$ is a compact subset of $X$.

Definition. A metric space is called sequentially compact if every sequence in $X$ has a convergent subsequence.

Definition. A metric space is called totally bounded if for every $\epsilon > 0$ there is a finite cover of $X$ consisting of balls of radius $\epsilon$.

THEOREM. Let $X$ be a metric space, with metric $d$. Then the following properties are equivalent (i.e. each statement implies the others):
(i) $X$ is compact.
(ii) $X$ has the Bolzano-Weierstrass property, namely that every infinite set has an accumulation point.*
(iii) $X$ is sequentially compact, i.e. every sequence has a convergent subsequence.
(iv) $X$ is totally bounded and complete.

*If $A$ is a subset of $X$ then $p$ is called an accumulation point if every neighborhood of $p$ contains a point $q \in A$ so that $q \neq p$. In Rudin’s book the terminology ‘limit point’ is used for this.
Example: A closed bounded interval $I = [a, b]$ in $\mathbb{R}$ is totally bounded and complete, thus compact. For the proof that $I$ is totally bounded note that we can cover $I$ with $N(\varepsilon)$ intervals of length $\varepsilon$ where $N(\varepsilon) \leq 10\varepsilon^{-1}(b - a)$.

Example: Any closed bounded subset of $\mathbb{R}^n$ is totally bounded and complete. For the proof note that any ball of radius $R$ (with respect to the usual Euclidean metric) can be covered that $N(\varepsilon)$ intervals where $N(\varepsilon) \leq (10dR)^d\varepsilon^{-d}$; you can obtain of course a somewhat better constant.

Example: Let $\mathfrak{B}$ be the metric space of all bounded sequences on $\mathbb{N}$, with metric $d(a, b) = \sup_{n \in \mathbb{N}} |a_n - b_n|$. Let $A$ be the closed ball of radius 1 centered at the zero sequence $(0, 0, \ldots)$. Then $A$ is bounded and closed but not sequentially compact (and by the theorem neither compact).

Indeed let $e^{(k)}$ be the member of $\mathfrak{B}$ with $e^{(k)}(n) = 0$ if $k \neq n$ and $e^{(n)}(n) = 1$. Then $d(e^{(k)}, e^{(l)}) = 1$ if $k \neq l$. Thus the $e^{(k)}$ form a bounded sequence in $\mathfrak{B}$ which does not have a convergent subsequence. The set $A$ is complete (as a closed subset of a complete space) but it is not totally bounded. Show this directly from the definition!

We conclude that merely bounded and complete sets in an arbitrary metric space may not be compact. Thus the cases of $\mathbb{R}^n$ and $\mathbb{C}^n$ (where this characterization of compact subsets holds) are exceptional instances which do not have ‘infinite dimensional’ analogues. The condition of total boundedness is crucial.

Example: Let $\ell^1$ denote the space of all absolutely summable sequences, i.e. the space of all sequences $\{a_n\}_{n=1}^{\infty}$ for which $\sum |a_n|$ converges. A metric in $\ell^1$ is given by $d_1(a, b) = \sum_{n=1}^{\infty} |a_n - b_n|$. Let $A$ be the closed ball of radius 1 centered at the zero sequence $(0, 0, \ldots)$ (i.e. the set of all absolutely summable sequences for which $\sum_{n=1}^{\infty} |a_n| \leq 1$.

Verify as in the previous example that $A$ is bounded and closed but not sequentially compact (and therefore not compact).

However one can show that the set of sequences $\{a_n\}_{n=1}^{\infty}$ for which $|a_n| \leq 2^{-n}$ for all $n \geq 1$ is a compact subset of $A$.

Example: Compact sets of continuous function (with respect to the sup metric) can be characterized by the Arzela-Ascoli theorem, see the section “equicontinuous families of functions” in ch. 7 of Rudin’s book.

Lemma 1. Any closed subset of a compact metric space is compact.
The proof of the main theorem is contained in a sequence of lemmata which we now state. In the subsequent sections we discuss the proof of the lemmata.

**Lemma 2.** A metric space is sequentially compact if and only if every infinite subset has an accumulation point.

**Lemma 3.** A compact metric space is sequentially compact.

**Lemma 4.** A sequentially compact subset of a metric space is bounded \(^1\) and closed.

**Lemma 5.** A metric space which is sequentially compact is totally bounded and complete.

**Lemma 6.** A metric space which is totally bounded and complete is also sequentially compact.

**Lemma 7.** A sequentially compact space is compact.

In what follows we shall always assume (without loss of generality) that the metric space \(X\) is not empty.

1. **Proof of Lemma 1:**
   **ANY CLOSED SUBSET OF A COMPACT METRIC SPACE IS COMPACT.**

   Let \(F\) be a closed subset of \(X\) and let \(\{G_\alpha\}_{\alpha \in A}\) be an open cover of \(F\). Every \(G_\alpha\) is of the form \(U_\alpha \cap F\) where \(U_\alpha\) is open in \(X\) (see Theorem 2.30 in Rudin’s book). Since \(X \setminus F\) is open in \(X\) as the complement of a closed set the collection \(\mathfrak{M}\) of sets consisting of the \(U_\alpha, \alpha \in A\) together with \(X \setminus F\) form an open cover of \(X\). As \(X\) is compact there is a finite subcover which consists of \(U_{\alpha_1}, \ldots, U_{\alpha_N}\), for suitable indices \(\alpha_1, \ldots, \alpha_N\), and possibly also \(X \setminus F\). But the latter set is disjoint from \(F\) and so the sets \(U_{\alpha_1}, \ldots, U_{\alpha_N}\) form a cover of \(F\). Since \(U_\alpha \cap F = G_\alpha\) the sets \(G_{\alpha_1}, \ldots, G_{\alpha_N}\) form a cover of \(F\) which is a subcover of the original collection. \(\square\)

2. **Proof of Lemma 2:**
   **A METRIC SPACE IS SEQUENTIALLY COMPACT IF AND ONLY IF EVERY INFINITE SUBSET HAS AN ACCUMULATION POINT.**

   Let \(Y\) be an infinite subset of \(X\) and let \(\{p_\alpha\}_{\alpha \in \mathbb{N}}\) be a sequence of pairwise different points in \(Y\). Since \(X\) is sequentially compact the sequence contains a convergent subsequence and the limit is an accumulation point of \(Y\).

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\(^1\) A set \(A\) in a metric space is bounded if the diameter \(\text{diam}(A) = \sup\{d(x, \tilde{x}) : x \in A, \tilde{x} \in A\}\) is finite. This is the same as saying that \(A\) is contained in a fixed ball (of finite radius).
Vice versa let $X$ be a metric space with the Bolzano-Weierstrass property, i.e. every infinite subset has an accumulation point. Now let $\{p_n\}$ be a sequence of points in $X$. If one point occurs an infinite number of times in the sequence then it has a ‘constant’ subsequence which of course converges. If every point in the sequence occurs only a finite number of times then we may choose a subsequence $\{p_{n_k}\}_{k\in\mathbb{N}}$ whose members are all pairwise different. But by assumption this set has an accumulation point and we can choose a subsequence which converges to this accumulation point. □

3. Proof of Lemma 3:
A compact metric space is sequentially compact.

By Lemma 2 we need to show the Bolzano-Weierstrass property, i.e. every infinite subset of $X$ has an accumulation point.

Suppose not, so let $Y$ be an infinite subset of $X$ which does not have an accumulation point. Then for every $y \in Y$ there is an open ball $B_y$ centered at $y$ such that $B_y$ contains no other points in $y$.

As $Y$ has no accumulation points $Y$ is closed in $X$ and, by Lemma 1, $Y$ is compact. For every $y \in Y$ the singleton set $\{y\} = B_y \cap Y$ is an open set in the metric space $Y$. Since $Y$ is infinite they form an open cover from which we cannot select an open subcover, which gives a contradiction (since $Y$ is compact). □

4. Proof of Lemma 4:
A sequentially compact subset of a metric space is bounded and closed.

Let $K$ be a compact subset of $X$. We first show that $K$ is bounded, i.e. the diameter of $K$

$$\text{diam}(K) = \sup \{d(x, \tilde{x}) : x \in K, \tilde{x} \in K\}$$

is finite.

Suppose it is not finite. Then for a fixed $y_1$ we can choose $y_2$ so that $d(y_1, y_2) \geq 1$. Since the diameter is not finite we can choose a point $y_3$ so that $d(y_1, y_3) \geq 1 + d(y_1, y_2)$ and we continue this way so that if for $n \geq 3$ points $y_1, \ldots, y_{n-1}$ so that $d(y_1, y_i) \geq 1 + d(y_1, y_{i-1})$ for $i \leq 3 \leq n$ we choose a point $y_n$ so that $d(y_1, y_n) \geq 1 + d(y_1, y_{n-1})$.

It is easy to see that this implies

$$d(y_1, y_m) \geq 1 + d(y_1, y_n) \text{ for } m > n .$$

Thus by the triangle inequality

$$d(y_m, y_n) \geq |d(y_m, y_1) - d(y_n, y_1)| \geq 1$$
and therefore the sequence \( \{y_n\} \) does not have a convergent subsequence. Thus the space is not sequentially compact and by Lemma 3 it is not compact, a contradiction to our hypothesis.

Thus we have shown that \( K \) is bounded. To prove that \( K \) is closed let \( \{p_n\} \) be a convergent sequence of points in \( K \); we have to show that the limit belongs to \( K \). But since \( K \) is sequentially compact this sequence has a subsequence which converges to a limit in \( K \). Thus the limit of the convergent sequence \( \{p_n\} \) belongs to \( K \). □

5. Proof of Lemma 5:
A sequentially compact metric space is totally bounded and complete

**Definition:** A set \( A \) is called an \( \varepsilon \)-net for \( X \) if \( A \) is finite and if the balls \( B_\varepsilon(x) \) (with radius \( \varepsilon \) and center \( x \)) where \( x \in A \), cover \( X \).

We consider a sequentially compact space \( X \) and let \( \varepsilon > 0 \).

Claim: Let \( \varepsilon > 0 \) and let \( A \subset X \) be a set of points of mutual distance \( \geq \varepsilon \) (i.e. if \( p \in A \) and \( q \in A \) and \( p \neq q \) then \( d(p,q) \geq \varepsilon \)). Then \( A \) is finite.

Suppose the claim is not true, then we can construct a sequence \( x_n \in X \) so that \( d(x_n,x_m) \geq \varepsilon \) whenever \( m \neq n \) and clearly this sequence does not have a convergent subsequence, in contradiction to the sequential compactness of \( X \).

We now construct a finite \( \varepsilon \)-net.

Pick a point \( p_1 \). Then (if possible) pick a point \( p_2 \) with \( d(p_1,p_2) \geq \varepsilon \), (if not possible, stop). Then (if possible) pick a point \( p_3 \) with \( d(p_1,p_3) \geq \varepsilon \) and \( d(p_2,p_3) \geq \varepsilon \), if not possible stop.

Continue (if possible) until points \( p_1,p_2,\ldots,p_m \) are chosen for which \( d(p_i,p_j) \geq \varepsilon \) for \( 1 \leq i < j \leq m \). Then pick a point \( p_{m+1} \) so that \( d(p_i,p_{m+1}) \geq \varepsilon \) for \( i = 1,\ldots,m \). If this is not possible then stop, and in this case every point in \( X \) is contained in an open ball of radius \( \varepsilon \) centered at one of the points \( p_1,\ldots,p_m \), so we have a finite \( \varepsilon \)-net of points.

By the claim above the construction stops after a finite number of steps, and the resulting set of points obtained before stopping form a finite \( \varepsilon \)-net for \( X \). Thus we have shown that \( X \) is totally bounded.

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Next we observe that \( X \) is complete. Let \( \{x_n\} \) be a Cauchy sequence in \( X \). Since \( X \) is sequentially compact this sequence has a convergent subsequence whose limit is also the limit of the Cauchy-sequence (provide the details or refer to a previously done exercise!). □

\(^1\)An alternative shorter formulation of this argument goes as follows: Let \( P \) be an \( \varepsilon \)-separated subset of \( X \) which is maximal with respect to inclusion. Then \( P \) is finite, by the above claim. The balls \( B_\varepsilon(p), p \in P \) form a finite \( \varepsilon \)-net.
6. Proof of Lemma 6:
A totally bounded and complete metric space is sequentially compact.

We consider a sequence \( \{a_n\} \) in \( X \) and assume that \( X \) is a totally bounded and complete metric space.

By the assumption ‘\( X \) totally bounded’ (applied with \( \varepsilon = 1 \)) we can cover the space \( X \) with finitely many balls of radius 1; then one of them contains \( a_n \)'s for infinitely many \( n \)'s; i.e. there is a ball \( B_1 \) of radius 1 so that there is a subsequence of \( \{a_n\} \) whose members all belong to \( B_1 \). We denote this subsequence by \( \{a_n^{(1)}\} \) and thus all \( a_n^{(1)} \) belong to \( B_1 \).

Similarly by the totally boundedness condition with \( \varepsilon = 1/2 \) we can find a subsequence \( \{a_n^{(2)}\} \) of \( \{a_n^{(1)}\} \) and a ball \( B_2 \) of radius \( 1/2 \) so that all \( a_n^{(2)} \) belong to \( B_2 \). Continuing in this way we obtain for any \( k \geq 2 \) a subsequence \( \{a_n^{(k)}\} \) of \( \{a_n^{(k-1)}\} \) and a ball \( B_k \) of radius \( 2^{-k} \) so that all \( a_n^{(k)} \) belong to \( B_k \).

Now consider the sequence \( \{a_n^{(n)}\} \) which is a subsequence of the original sequence. We show that it is a Cauchy-sequence. Indeed if \( m \geq n \) we have by the triangle inequality
\[
d(a_m^{(m)}, a_n^{(n)}) \leq d(a_m^{(m)}, a_m^{(m-1)}) + \cdots + d(a_n^{(n+1)}, a_n^{(n)})
\]
and since \( a_j^{(j)} \) and \( a_j^{(j-1)} \) are both in \( B_{j-1} \) their mutual distance is \( \leq 2 \cdot 2^{1-j} \). Thus the previous displayed inequality implies that for \( m > n \)
\[
d(a_m^{(m)}, a_n^{(n)}) \leq 2^{2-m} + \cdots + 2^{2-(n+1)} \leq 2^{2-n}
\]
which shows that \( \{a_n^{(n)}\} \) is a Cauchy sequence. Thus by the assumed completeness it converges, and we have found a convergent subsequence of \( \{a_n\} \). \( \square \)

7. Proof of Lemma 7:
A sequentially compact space is compact

To show this we first prove an auxiliary statement:

**Sublemma.** Let \( X \) be a sequentially compact space. Suppose we are given an infinite open cover \( \{G_\alpha\}_{\alpha \in A} \) of \( X \). Then there exists an \( \varepsilon > 0 \) so that every ball of radius \( \varepsilon \) is contained in one of the (open) sets \( G_\alpha \).

We argue by contradiction and assume that the statement does not hold. Then for every \( n \in \mathbb{N} \) there is a ball \( B_n \) of radius \( 1/n \) which is not contained in any of the sets \( G_\alpha \). Let \( p_n \) be the center of \( B_n \). Since we assume that \( X \) is sequentially compact the sequence of centers has a convergent subsequence \( \{p_n_k\} \) whose limit we denote by \( p \). Since the \( G_\alpha \) is a cover there is an index \( \alpha_\alpha \) so that \( p \in G_\alpha_\alpha \). As \( p \) is an interior point of the (open) set \( G_\alpha_\alpha \) it contains an open ball of radius \( \delta > 0 \). Also there is an \( M \) so that for \( k \geq M \) we have \( d(p_{n_k}, p) < \delta/2 \). By the triangle inequality we see that the ball \( B_{n_k} \) is contained in \( G_\alpha_\alpha \).
provided that $k > M$. But this is a contradiction to the construction of the sequence \( \{p_n\} \) (which implied that none of the balls $B_n$ is contained in any of the sets $G_\alpha$).

We now proceed to show that a sequentially compact is compact. We need to show that a given open cover \( \{G_\alpha\}_{\alpha \in A} \) of $X$ contains a finite subcover. By the sublemma there exists an $\varepsilon > 0$ so that every ball of radius $\varepsilon$ is contained in one of the (open) sets $G_\alpha$. We have shown in Lemma 5 that a sequentially compact space is totally bounded; thus there exist points \( \{p_1, \ldots, p_k\} \) so that $X$ is contained in the union of the balls $B_\varepsilon(p_i)$, $i = 1, \ldots, k$. As each $B_\varepsilon(p_i)$ is contained in one of the sets in the cover, say in $G_\alpha_i$, the collection $G_\alpha_i$, $i = 1, \ldots, k$ is a finite subcover.

Exercises:

1. Prove that a totally bounded metric space is separable (i.e. contains a countable dense subset).

2. A collection \( \{F_\alpha : \alpha \in A\} \) of closed sets has the finite intersection property if for every finite subset $A_o$ of $A$ the intersection $\cap_{\alpha \in A_o} F_\alpha$ is not empty.

   Prove that the following statements (i), (ii) are equivalent.

   (i) A metric space $X$, with metric $d$, is compact.

   (ii) For every collection \( \{F_\alpha\}_{\alpha \in A} \) of closed sets with the finite intersection property it follows that

   \[
   \bigcap_{\alpha \in A} F_\alpha \neq \emptyset.
   \]

3. Let $\ell^1$ denote the space of all absolutely summable sequences, i.e. the space of all sequences $\{a_n\}_{n=1,2,\ldots}$ for which $\sum |a_n|$ converges, with the metric $d(a, b) = \sum_{n=1}^{\infty} |a_n - b_n|$.

   (i) Prove that the set of all sequences $\{a_n\}$ which satisfy $|a_n| \leq 2^{-n}$ for all $n \in \mathbb{N}$ is compact.

   (ii) More generally, if $\{b_n\}$ is a fixed sequence of nonnegative terms with the property that $\sum b_n < \infty$ then the set of all sequences $\{a_n\}$ which satisfy $|a_n| \leq b_n$ is compact.

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