Metric entropy and covering numbers. Let $E$ be a totally bounded subset of a metric space $X$, i.e. for every $\delta > 0$ it is contained in a finite collection of $\delta$-balls.

For $\delta > 0$ let $N(E, \delta)$ be the minimal number of $\delta$-balls needed to cover $E$ (the centers of these balls are not required to belong to $E$). This number is called the $\delta$-covering number of $E$; note that it depends not only on $E$ but also on the underlying metric space $X$ and the given metric $d$. The function $\delta \mapsto \log N(E, \delta)$ is called the metric entropy function of $E$.

One is interested in the behavior of $N(E, \delta)$ for small $\delta$. For compact $E$ this serves as a quantitative measure of compactness.

We also set $N(E, \delta) = \infty$ if $E$ is not totally bounded.

The number $\dim(E) = \limsup_{\delta \to 0^+} \frac{\log N(E, \delta)}{\log(1/\delta)}$ is called the upper Minkowski dimension or upper metric dimension of $E$.

The analogous expression $\dim(K) = \overline{\dim}(E) = \alpha$ where the lim sup is replaced by a lim inf is called lower Minkowski dimension or lower metric dimension of $E$. If $\dim(K) = \overline{\dim}(E) = \alpha$ we say that $E$ has Minkowski dimension $\alpha$.

1. (i) Show that if we replace the natural log in the above definitions by another log$_b$ with base $b > 1$ then the definitions of the dimensions do not change.

(ii) Let $\alpha > 0$. Suppose that for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ and a positive constant $C_\varepsilon \geq 1$ such that $C_\varepsilon^{-1} \delta^{-\alpha-\varepsilon} \leq N(E, \delta) \leq C_\varepsilon \delta^{-\alpha-\varepsilon}$ for $0 < \delta < \delta(\varepsilon)$. Show that $E$ has Minkowski dimension $\alpha$.

(iii) Let $E \subset X$ be totally bounded and let $\overline{E}$ be the closure of $E$. Then $\overline{E}$ is totally bounded and we have

$$N(E, \delta) \leq N(\overline{E}, \delta) \leq N(E, \delta') \text{ if } 0 < \delta' < \delta.$$ 

(iv) Define $N^{\text{cent}}(E, \delta)$ to be the minimal number of $\delta$-balls with center in $E$ needed to cover $E$. Show that

$$N(E, \delta) \leq N^{\text{cent}}(E, \delta) \leq N(E, \delta/2).$$

(v) Let $B_1, \ldots, B_M$ be balls of radius $\delta$ in $X$, so that each ball has nonempty intersection with the set $E$. For each $i = 1, \ldots, M$ denote by $B^*_i$ the ball with same center as $B_i$ and radius $3\delta$. Assume that the balls $B^*_1, \ldots, B^*_M$ are disjoint. Prove that $M \leq N(E, \delta)$.

Remark: This can be an effective tool to prove lower bounds for the covering numbers.
2. Consider the following norms in $\mathbb{R}^n$

$$
\|x\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad \|x\|_\infty = \max_{i=1,\ldots,n} |x_i|.
$$

with associated metrics $d_1, d_2, d_\infty$.

(i) Recall that

$$
\|x\|_\infty \leq \|x\|_1 \leq \sqrt{n} \|x\|_2 \leq n \|x\|_\infty
$$

for all $x \in \mathbb{R}^n$.

(ii) Let $E \subset \mathbb{R}^n$ and let $N_1(E, \delta), N_2(E, \delta), N_\infty(E, \delta)$ be the metric entropy numbers of $E$ associated with to the metrics $d_1, d_2, d_\infty$, respectively. Show that

$$
N_\infty(E, \delta) \leq N_2(E, \delta) \leq N_1(E, \delta) \leq N_2(E, \delta/\sqrt{n}) \leq N_\infty(E, \delta/n).
$$

(iii) Let $Q = [0,1]^n$ be the unit cube in $\mathbb{R}^n$. Show that $Q$ has Minkowski dimension $n$ (with respect to any of the metrics $d_1, d_2, d_3$).

(iv) Let $f$ be a differentiable function on $[0,1]$ with bounded derivative. Let $E$ be the set of all $x = (x_1, x_2) \in \mathbb{R}^2$ for which $0 \leq x_1 \leq 1$ and $x_2 = f(x_1)$. What is the Minkowski dimension of $E$?

(v) Recommended only exercise for those of you who know the Cantor middle third set: its Minkowski dimension is equal to $\log 2/\log 3$.

3. (i) Let $\beta > 0$. Consider the subset $E$ of $\mathbb{R}$ consisting of the numbers $n^{-\beta}$, for $n = 1, 2, \ldots$. Show that $E$ has a Minkowski dimension and determine it.

*Hint:* It might help to try this first for the sequence $1/n$ which, perhaps counterintuitively, turns out to have Minkowski dimension $\frac{1}{2}$.

(ii) Recommended only exercise for those of you who know the Cantor middle third set: its Minkowski dimension is equal to $\frac{\log 2}{\log 3}$.

4. Let $A$ be the space of functions $f : \mathbb{N} \to \mathbb{R}$ (aka sequences) so that $|f(n)| \leq 2^{-n}$ for all $n \in \mathbb{N}$. It is a subset of the space of bounded sequences with norm $\|f\|_\infty = \sup_{n \in \mathbb{N}} |f(n)|$ and associated metric $d_\infty$. Show that for $\delta < 1/2$ the covering numbers $N(A, \delta)$ satisfy the bounds

$$
N(A, \delta) \leq \left(\frac{1}{\delta}\right)^{C + \frac{1}{2} \log_2 \frac{1}{\delta}}
$$

where $C$ is independent of $\delta$. *Hint:* It helps to work with $\delta = 2^{-M}$ where $M \in \mathbb{N}$.

Also provide a lower bound which shows that $A$ does not have finite lower Minkowski dimension.

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*Skip this - it was proved in class*