LEBESGUE SPACE ESTIMATES FOR A CLASS OF FOURIER
INTEGRAL OPERATORS ASSOCIATED WITH WAVE PROPAGATION

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Dedicated to Professor Hans Triebel

Abstract. We prove $L^q$ estimates related to Sogge’s conjecture for a class of Fourier
integral operators associated with wave equations.

1. Introduction

In this note we prove a variable coefficient version of a recent result in [6] on the local
$L^q$ space-time regularity results for solutions of wave equations. The solution operators are
Fourier integral operators satisfying the ‘cinematic curvature’ hypothesis introduced in [17]
(see also [14]).

For the general setup let $Y$ and $Z$ be paracompact $C^\infty$ manifolds, $\dim(Y) = d$, $\dim(Z) =
d + 1$; in the current paper we shall need to assume $d \geq 4$. We are interested in sharp local
regularity estimates for Fourier integral operators $F \in I^{\mu-1/4}(Z, Y; \mathcal{C})$ (associated with the
Fourier integral distributions defined in [8]). Here the canonical relation

$$\mathcal{C} \subseteq T^*Z \setminus 0_L \times T^*Y \setminus 0_R$$

is a conic manifold of dimension $2d + 1$, which is Lagrangian with respect to the symplectic
form $d\zeta \wedge dz - d\eta \wedge dy$. We denote by $0_L$ and $0_R$ the zero-sections in $T^*Z$ and $T^*Y$,
respectively.

We formulate a curvature hypothesis which appeared in [3], [10] for classes of oscillatory
integral operators (see [13], [2] for current results on these classes). We follow the exposition
in [14] and impose conditions on the following projection maps.

$$\begin{array}{ccc}
\mathcal{C} & \uparrow & \downarrow \\
T^*Y \setminus 0 & \downarrow & Z \\
& \downarrow & T^*_zZ \setminus 0
\end{array}$$

We require that the projection $\pi_L : \mathcal{C} \to T^*Y$ is a submersion (i.e. the differential has
maximal rank $2d$). We also require that the space projection $\Pi_Z : \mathcal{C} \to Z$ is a submersion
(i.e. its differential has maximal rank $d + 1$). As discussed in §2 of [14] this implies that
for fixed $z \in \Pi_Z\mathcal{C}$ the image of the projection to the fiber,

$$\Gamma_z = \{\zeta : \in T^*_zZ : \exists (y, \eta) \text{ such that } (z, \zeta, y, \eta) \in \mathcal{C}\},$$

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is an immersed conic \((d - 1)\)-dimensional hypersurface in \(T^*_z \mathcal{Z} \setminus 0_L\). We then make an assumption on the curvature of the cones \(\Gamma_z\):

**Curvature hypothesis** \(\mathcal{H}(\ell)\): \(\mathcal{C} \subset T^* \mathcal{Z} \setminus 0_L \times T^* \mathcal{Y} \setminus 0_R\), the projections \(\pi_R\) and \(\Pi_Z\) are submersions and for each \(z\) the cone \(\Gamma_z\) has at least \(\ell\) nonvanishing principal curvatures at any \(\zeta \in \Gamma_z\).

**Theorem 1.1.** Let \(\ell \geq 3\), let \(\mathcal{C}\) satisfy hypothesis \(\mathcal{H}(\ell)\) and let \(\mathcal{F} \in I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})\). Suppose \(\frac{2\ell}{d-2} < q < \infty\) and \(\mu < \frac{d-1}{4} - \frac{1}{q}\). Then \(\mathcal{F}\) maps \(L^q_{\text{comp}}(\mathcal{Y})\) to \(L^q_{\text{loc}}(\mathcal{Z})\).

We may apply the theorem with \(\ell = d - 1\) to solutions of the wave equation on a compact Riemannian manifold \(M\), with initial data in \(L^q\)-Sobolev spaces \(L^q_0(M)\). Let \(\Delta\) be the Laplace-Beltrami operator on \(M\). If one combines Theorem 1.1 with the usual parametrix construction (cf. [4]) one obtains (arguing as in [14])

**Corollary 1.2.** Let \(d \geq 4\), \(\frac{2(d-1)}{d-3} < q < \infty\), and let \(I\) be a compact time interval. There is \(C > 0\) such that

\[
\left( \int_I \|e^{it\sqrt{-\Delta}}f\|_{L^q(M)}^q \, dt \right)^{1/q} \leq C\|f\|_{L^q_0(M)}, \quad \alpha = \frac{d-1}{2} - \frac{d}{q},
\]

for all \(f \in L^q_0(M)\).

Note that the constant may strongly depend on the choice of \(I\). There are further regularity improvements in the scale of Triebel-Lizorkin spaces (cf. §3 below); in particular \(L^q_0(M)\) can be replaced by the Besov space \(B^q_0(M)\).

In §2 we prove a frequency localized version of Theorem 1.1 and combine the estimates corresponding to different frequencies in §3. In §4 we discuss some generalizations in the constant coefficient case.

**Remarks.** In the constant coefficient case one can recover from Theorem 1.1 the space time estimates of [6] which correspond to an endpoint version of Sogge’s conjecture in the range given for \(\ell = d - 1\), see also §4 for other generalizations. For previous partial results on Sogge’s conjecture, also in lower dimensions, see the groundbreaking paper of Wolff [20] and the subsequent papers [12], [5].

The case \(\ell = d - 1\) essentially corresponds to the assumption of cinematic curvature in [17]. We use Hörmander’s convention for the definition of order, i.e., in view of the different dimensions of \(\mathcal{Z}\) and \(\mathcal{Y}\) operators of class \(I^{\mu-1/4}(\mathcal{Z}, \mathcal{Y}; \mathcal{C})\) correspond to locally finite sums of operators with integral kernels in the standard representation (1) below, involving \(d\) frequency variables and standard symbols of order \(\mu\). One can use a partition of unity and finite decompositions in the fiber variable to reduce matters to the estimation of an integral operator with compactly supported kernel \(\mathcal{K}\) which is given as an oscillatory integral distribution in the sense of [8]. Namely if \(Z\) is an open set in \(\mathbb{R}^{d+1}\) and \(Y\) is an open set in \(\mathbb{R}^d\) we may assume that

\[
(1) \quad \mathcal{K}(z, y) = \int a(z, y, \xi)e^{i(\phi(z, \xi) - (y, \xi))} \, d\xi
\]

where \(a\) is a standard symbol of order \(\mu\), \(a\) is supported for \(z, y\) in compact subsets of \(Z\) and \(Y\), resp., and \(\phi\) is smooth away from the origin and homogeneous of order one with respect to the variable \(\xi\), and supported in an open set which is conic in \(\xi\). We then have \(\nabla_z \phi(z, \xi) \neq 0\) for \(\xi \neq 0\) and the mixed second derivative \((d + 1) \times d\) matrix \(\phi''_{\xi\xi}(z, \xi)\) has rank \(d\). For fixed \((z, \xi)\), if the vector \(u\) is in the cokernel of \(\phi''_{\xi\xi}(z, \xi)\) then the Hessian matrix \(\nabla_{\xi\xi}(u, \nabla_z \phi)(z, \xi)\) has rank at least \(\ell\), by our curvature assumption.
2. The frequency localized case

By making further localizations, changing variables in \( z \) and \( y \), and ignoring error terms which are smoothing of high order we may assume that our kernel is given by

\[
K(z, y) = \sum_{k=1}^{\infty} 2^{k\ell} K_k(z, y),
\]

where

\[
K_k(z, y) = \int \chi_k(z, y, 2^{-k} \xi) e^{i(\varphi(z, \xi) - (y, \xi))} d\xi,
\]

(2)

and the functions \( \chi_k \) are smooth and supported in a compact subset of \( Z \times Y \times \Xi \). Here \( Z \) is a small neighborhood of the origin in \( \mathbb{R}^{d+1} \), \( Y \) is a small neighborhood of the origin in \( \mathbb{R}^d \) and \( \Xi \) is a small neighborhood of the vector \( e_1 := (1, 0, \ldots, 0) \) in \( \mathbb{R}^d \). Moreover

\[
\varphi''_{z, \xi}(0, e_1) = \begin{pmatrix} I_d \cr 0 \end{pmatrix}
\]

(3)

\[
\text{rank } \nabla^2_{\xi \xi} \varphi'_{z, d+1}(0, e_1) \geq \ell;
\]

and in view of the small choice of \( Z, Y, \Xi \) we may assume that for all \( (z, \xi) \in Z \times \Xi \) the gradient \( \varphi'_{z}(z, \xi) \) is close to \( e_1 \), and, with \( z = (z', z_{d+1}) \), we may assume that \( \varphi''_{z, \xi}(z, \xi) \) is close to the identity matrix \( I_d \) and \( \varphi''_{z, d+1}(z, \xi) \) is small. We may further perform a rotation and assume that in coordinates \( \xi = (\xi_1, \xi', \xi'') \) with \( \xi' = (\xi_2, \ldots, \xi_{\ell+1}) \) we have

\[
\text{rank } \nabla^2_{\xi' \xi' \xi''_{z, d+1}}(0, e_1) = \ell.
\]

(5)

Finally, \( |\partial_{z, y, \xi}^\alpha \chi_k(z, y, \xi)| \leq C_\alpha \) for any multiindex \( \alpha \), uniformly in \( k, (z, y, \xi) \in Z \times Y \times \Xi \).

Let \( T_k f(z) = \int K_k(z, y) f(y) dy \). Here we prove that the \( L^q(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^{d+1}) \) operator norm of \( T_k \) is \( O(2^{k(d+1)q - \frac{d}{2}}) \) for \( q > \frac{2\ell}{d+2} \), and in the next section we discuss how to put the estimates for \( T_k \) together. The \( L^\infty \) estimate

\[
|T_k f|_{L^\infty} \lesssim 2^{k \frac{d+1}{2}} |f|_{L^\infty}
\]

can be found in [16]. By interpolation it is enough to prove

**Theorem 2.1.** Let \( \ell \geq 3 \) and \( q_\ell = \frac{2\ell}{d+2} \). The operator \( T_k \) is of restricted weak type \((q_\ell, q_\ell)\), with operator norm

\[
|T_k|_{L^{q_\ell, 1}(\mathbb{R}^{d+1}) \rightarrow L^{q_\ell, \infty}(\mathbb{R}^d)} \lesssim 2^{k(d/\ell - 1/2)}.
\]

By duality we need to prove the restricted weak type inequality for the adjoint operator \( T_k^* \), given by

\[
T_k^* F(y) = \int \int \chi_k(z, y, 2^{-k} \xi) e^{i(y, \xi) - \varphi(z, \xi)} d\xi F(z) dz,
\]

i.e. for each measurable set \( E \) contained in \([-1/2, 1/2]^{d+1}\),

\[
|T_k^* \chi_E|_{L^{p_\ell, \infty}} \lesssim 2^{k \frac{d+4}{\ell+2}} |E|, \quad p_\ell = \frac{2\ell}{\ell + 2}.
\]

The estimate (6) will be derived from the following Proposition 2.2, which is a discretized version of (6) and will be proved in §2.3.
Proposition 2.2. Let $p_k = \frac{2^k}{1 + 2^k}$, $\ell \geq 3$. For $k > 2$ let $Z_k = 2^{-k}\mathbb{Z}^{d+1} \cap [-\varepsilon^2, \varepsilon]^d$, for sufficiently small $\varepsilon > 0$. Suppose that for each $z \in Z_k$ we are given a symbol $a_{k,3}$ supported in $\{\xi : 2^{k-1} < |\xi| < 2^{k+1}, |\xi| - e_1 \leq \varepsilon^2\}$ so that

\[
|\partial_{\xi}^\alpha a_{k,3}(\xi)| \leq 2^{-k|\alpha|}, \quad |\alpha| \leq 10d.
\]

Define $S_z \equiv S_z^k$ by

\[
S_z(y) = \int a_{k,3}(\xi) e^{i(\langle y, \xi \rangle - \varphi(y, \xi))} d\xi.
\]

Then for each $E \subset Z_k$ we have

\[
\text{meas}\left(\left\{y \in \mathbb{R}^d : \sum_{z \in E} S_z \right\} > \alpha\right) \leq C 2^k(\frac{d+1}{2} - 1) \alpha^{-\ell} \#E.
\]

In the following subsections we prove some preparatory $L^1$ and $L^2$ estimates, then prove Proposition 2.2, and that Proposition 2.2 implies (6). In §3 we combine the dyadic estimates in Theorem 2.1.

2.1. $L^1$ estimates. $L^1$-estimates for the expressions $S_z$ can be found in [16]. In what follows we let $\Theta_k$ be a maximal $2^{-k/2}$-separated set of unit vectors. Using a homogeneous extension of a partition of unity on the sphere one can split

\[
a_{k,3}(\xi) = \sum_{\theta \in \Theta_k} a_{k,3,\theta}(\xi)
\]

where $a_{k,3,\theta}$ is supported on the intersection of the cone $\{\xi : |\xi| - \theta \leq 2^{-k/2}\}$ with the support of $a_{k,3}$; moreover if $u_i$ are unit vectors perpendicular to $\theta$ we have the estimates

\[
|\langle \theta, \nabla \xi \rangle^{M_i} \prod_{i=1}^{M_2} \langle u_i, \nabla \xi \rangle a_{k,3,\theta}(\xi)| \leq C(M_1, M_2) 2^{-kM_1} 2^{-kM_2/2}
\]

whenever $M_1 + M_2 \leq 10d$. Let

\[
S_{3,\theta}(y) = \int a_{k,3,\theta}(\xi) e^{i(\langle y, \xi \rangle - \varphi(y, \xi))} d\xi.
\]

By homogeneity we have

\[
\phi_{\xi\xi}(z, \theta) = 0.
\]

Using this observation we get, as in [16], by an integration by parts

\[
|S_{3,\theta}(y)| \leq C_d 2^k \frac{d+1}{2} \left(1 + 2^k |\varphi_{\xi}^\prime(3, \theta) - y, \theta| + 2^{k/2} |\Pi_{\theta} \varphi_{\xi}^\prime(3, \theta) - y|\right)^{-10d};
\]

here $\Pi_{\theta}$ denotes the projection to the orthogonal complement of $\theta$. This estimate implies $\|S_{3,\theta}\|_1 = O(1)$ and therefore

\[
\|S_{3,\theta}\|_1 \lesssim 2^k \frac{d+1}{2}.
\]

Moreover we get for $1 \leq R \leq 2^k$,

\[
\int_{|\Pi_{\theta} \varphi_{\xi}^\prime(3, \theta) - y| \geq (2^{-k} R)^{1/2}} |S_{3,\theta}(y)| \, dy \lesssim \int_{|w'| \geq (2^{-k} R)^{1/2}} 2^{k-1/2} \left(1 + 2^{k/2} |w'|\right)^{10d-2} \, dw' \lesssim R^{1-9d}.
\]
and similarly
\[ \int_{|e_{y}(\bar{z},\bar{\theta})| \geq 2^{-k} R} |S_{y}(y)| \, dy \leq \int_{|w_{y}| \geq 2^{-k} R} \frac{2^{k}}{(1 + 2^{k}|w_{y}|)^{9d-1}} \, dw_{y} \lesssim R^{2-9d}. \]

Now clearly
\[ |\varphi'_{\xi}(z, \theta) - \varphi'_{\xi}(\bar{z}, \bar{\theta})| \lesssim |z - \bar{z}| + |\theta - \bar{\theta}|, \]
and by (11) also
\[ |\langle \varphi'_{\xi}(z, \theta) - \varphi'_{\xi}(\bar{z}, \bar{\theta}), \theta \rangle| \lesssim |z - \bar{z}| + |\theta - \bar{\theta}|^2. \]

Thus if
\[ (13) \quad V_{\theta}^{k}(z, R) = \{ y : |\langle \varphi'_{\xi}(z, \theta) - y, \theta \rangle| \leq R^{2-k}, \quad |\Pi_{\theta}(\varphi'_{\xi}(z, \theta) - y)| \leq (R^{2-k})^{1/2} \} \]
then the above calculations give
\[ (14) \quad \|S_{y}\|_{L^1(R^d \setminus V_{\theta}^{k}(y_{0}, R))} \leq C(1) R^{-4d} \quad \text{if } |\phi - y| \leq C_{1} R^{-k}, \quad |\theta - \phi| \leq C_{1}(R^{2-k})^{1/2} \]
for \( C_{1} \geq 1. \)

2.2. Estimates for scalar products. Based on standard calculations for oscillatory integrals ([10], [18], [1], [11], [14]) we prove some estimates for scalar products \( \langle S_{y}, S_{y}' \rangle \); these results are closely related to the scalar product estimates in [6]. For the Fourier transforms we have
\[ \tilde{S}_{y}(\xi) = a_{k,\theta}(\xi)e^{-i\varphi(\xi, \theta)} \]
and
\[ (15) \quad (2\pi)^{d}\langle S_{y}, S_{\bar{y}} \rangle = \langle \tilde{S}_{y}, \tilde{S}_{\bar{y}} \rangle = \int_{a_{k,\theta}(\eta)a_{k,\theta}(\eta)}^{a_{k,\theta}(\eta)a_{k,\theta}(\eta)} e^{i\langle \varphi(\xi, \eta) - \varphi(\xi, \eta) \rangle} \, d\eta \]
\[ = 2^{kd} \int_{b_{k,\theta}} b_{k,\theta}(\xi)e^{i2k\langle \varphi(\xi, \eta) - \varphi(\xi, \eta) \rangle} \, d\xi \]
where \( b_{k,\theta} \) is supported on a subset of diameter \( O(\varepsilon^2) \) of the annulus \( \{ |\xi| \approx 1 \} \), near \( e_{1} \), with \( \varepsilon \) sufficiently small. We may assume in what follows that \( \phi, \bar{\phi} \) are in a neighborhood of the origin in \( \mathbb{R}^{d+1} \), of diameter \( \lesssim \varepsilon^{2} \). We split coordinates \( z = (z', z_{d+1}) \), take advantage of (3) and get
\[ |\varphi'_{\xi}(\xi, \phi) - \varphi'_{\xi}(\xi, \phi)| \geq c|\phi' - \phi'| - C_{1}|\phi'_{d+1} - \phi'_{d+1}| \]
and after an integration by parts we get
\[ (17) \quad \langle S_{y}, S_{\bar{y}} \rangle \lesssim \frac{2^{kd}}{(1 + 2^{k}|\phi - \phi'|)^{9d}} \quad \text{if } |\phi'_{d+1} - \phi'_{d+1}| \geq c|\phi'_{d+1} - \phi'_{d+1}| \]
For \( s \in [0, 1] \) set \( z_{s} = z + s(\phi - \bar{\phi}) \). If
\[ |\phi'_{d+1} - \phi'_{d+1}| \leq C_{2}|\phi'_{d+1} - \phi'_{d+1}| \]
(with suitable \( C_{1} \ll C_{2} \ll \varepsilon^{-1} \)) we consider
\[ \varphi(\xi, \phi) - \varphi(\xi, \phi) = \int_{0}^{1} \left[ \varphi'_{s_{d+1}}(\phi, \xi) + \frac{\phi'_{d+1}}{\phi'_{d+1} - \phi'_{d+1}} \right] ds. \]
Note that \( \varphi(\xi, \phi) - \varphi(\xi, \phi) \) is a small perturbation of \( \varphi'_{s_{d+1}}(0, \xi) \) if \( \varepsilon \) is sufficiently small. We apply the method of stationary phase (with parameters, [8]) in the \( \xi' \)-variables, using (5). This yields
\[ \langle S_{y}, S_{\bar{y}} \rangle \lesssim \frac{2^{kd}}{(1 + 2^{k}|\phi - \phi'|)^{9d/2}} \quad \text{if } |\phi'_{d+1} - \phi'_{d+1}| \leq C_{2}|\phi'_{d+1} - \phi'_{d+1}|. \]
and combining this with (17) we get

\[ |\langle S_j, S_j' \rangle| \lesssim \frac{2^{kd}}{(1 + 2^k|\overline{j} - \overline{j}'|)^{\ell/2}} \]

whenever \(|\overline{j} - \overline{j}'| = O(\varepsilon^2)\).

2.3. Proof of Proposition 2.2. If \(\alpha \leq 2^{kd+1} \) then the desired inequality follows from (12). Indeed by Tshebyshev’s inequality the left hand side of (9) is \(\lesssim \alpha^{-1}2^{k(d-1)/2}\#E\)
which is dominated by the right hand side of (9) if \(\alpha \leq 2^{kd+1} \).

In what follows we shall therefore assume that \(\alpha > 2^{kd+1} \) and set

\[ u_k(\alpha) := (\alpha 2^{-k(d+1)})^{p_\ell} > 1. \]

The argument is a variant of one in [6]; it is based on a Calderón-Zygmund type decom-
position at height \(u_k(\alpha)\) where volume is replaced by diameter.

By the usual Vitali procedure there is a finite (possibly empty) family \(\mathcal{B}^k\) of disjoint balls so that

\[ u_k(\alpha)2^k \text{diam}(B) \leq \#(E \cap B) \quad \text{for } B \in \mathcal{B}^k; \]

moreover if we remove the balls in \(\mathcal{B}^k\) and set

\[ E_* = E \setminus \bigcup_{B \in \mathcal{B}^k} B, \]

then

\[ \#(E_* \cap B) \leq C_d u_k(\alpha)2^k \text{diam}(B) \quad \text{for every ball } B. \]

Since \(E \subset \mathcal{Z}_k\) which is \(2^{-k}\)-separated, we may assume that \(\text{diam}(B) \geq 2^{-k}\) if \(B \in \mathcal{B}^k\).

We need to establish the following two inequalities:

\[ \text{meas}\left(\left\{ y \in \mathbb{R}^d : \left| \sum_{B \in \mathcal{B}^k, j \in E \cap B} S_j \right| > \alpha/2 \right\} \right) \leq C2^{k(d+1)(p_\ell-1)}\alpha^{-p_\ell}\#E, \]

\[ \text{meas}\left(\left\{ y \in \mathbb{R}^d : \left| \sum_{j \in E_*} S_j \right| > \alpha/2 \right\} \right) \leq C2^{k(d+1)(p_\ell-1)}\alpha^{-p_\ell}\#E. \]

Proof of (22). We first form an exceptional set as follows. Let \(z_B\) denote the center of a ball \(B \in \mathcal{B}^k\) and let \(R_B = 10d2^k\text{diam}(B) \geq 1\). Let \(\Theta(k, B)\) be a maximal \(C_1(2^{-k}R_B)^{1/2}\) separated subset of \(S^{d-1}\). Here \(C_1\) is the constant in (14). Define (using the notation in (13))

\[ \mathcal{V}^k = \bigcup_{B \in \mathcal{B}^k} \bigcup_{q \in \Theta(k, B)} V^k_{q}(z_B, R_B). \]

Observe that \(\text{meas}(V^k_{q}(z_B, R_B)) = O((R_B2^{-k})^{(d+1)/2})\) and \(\#\Theta(k, B) = O((2^kR_B^{(d-1)/2})\). Thus

\[ \text{meas}(\mathcal{V}^k) \lesssim \sum_{B \in \mathcal{B}^k} \sum_{q \in \Theta(k, B)} \text{meas}(V^k_{q}(z_B, R_B)) \lesssim \sum_{B \in \mathcal{B}^k} R_B2^{-k} \]

\[ \lesssim \sum_{B \in \mathcal{B}^k} \text{diam}(B) \lesssim \sum_{B \in \mathcal{B}^k} 2^{-k}\#(E \cap B) \lesssim 2^{k(-1+(d+1)p_\ell)\alpha^{-p_\ell}\#E}, \]

by the disjointness of the balls in \(\mathcal{B}^k\), (20), and the definition of \(u_k(\alpha)\).
To conclude the proof of (22) we have to estimate the contribution in the complement of $\mathcal{V}^k$. For this we bound
\[
\text{meas}\left(\left\{ y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left| \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} S_j \right| > \frac{\alpha}{2} \right\} \right) \lesssim \alpha^{-1} \left\| \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} S_j \right\|_{L^1(\mathbb{R}^d \setminus \mathcal{V}^k)}.
\]

Now fix $B$. For every $\theta \in \Theta_k$ we may choose a $\vartheta = \vartheta_B(\theta) \in \Theta(k, B)$ so that $|\vartheta_B(\theta) - \theta| \leq C_1(R_B 2^{-k})^{-1/2}$. Recalling $S_j = \sum_{\theta \in \Theta_k} S_{j, \theta}$, we see
\[
\left\| \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} S_j \right\|_{L^1(\mathbb{R}^d \setminus \mathcal{V}^k)} \leq \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} \left\| S_{j, \vartheta} \right\|_{L^1(\mathbb{R}^d \setminus \mathcal{V}^k_{\vartheta_B(\theta)(z_B, R_B)})} \\
\lesssim \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} \sum_{\theta \in \Theta_k} R_B^{-4d} \lesssim \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} 2^{k(d-1)/2} R_B^{-4d}.
\]

For the second inequality we use (14) and the last one follows from $\#\Theta_k = O(2^{k(d-1)/2})$. Now we note that $2^{k(d-1)/2} \alpha^{-1} = 2^{k(-1 + \frac{d+1}{2} p)} \alpha^{-p} u_k(\alpha)^{1 - \frac{1}{p'}}$. Thus
\[
\text{meas}\left(\left\{ y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left| \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} S_j \right| > \frac{\alpha}{2} \right\} \right) \lesssim \alpha^{-1} \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} 2^{k(d-1)/2} R_B^{-4d} \\
\lesssim 2^{k(-1 + \frac{d+1}{2} p)} \alpha^{-p} u_k(\alpha)^{1 - \frac{1}{p'}} \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} R_B^{-4d}.
\]

By (20) we have for $B \in \mathfrak{B}^k$
\[
u_k(\alpha) \lesssim \frac{\#(\mathcal{E} \cap B)}{R_B} \lesssim R_B^d
\]
and therefore
\[
\text{meas}\left(\left\{ y \in \mathbb{R}^d \setminus \mathcal{V}^k : \left| \sum_{B \in \mathfrak{B}^k} \sum_{j \in \mathcal{E} \cap B} S_j \right| > \frac{\alpha}{2} \right\} \right) \lesssim 2^{k(-1 + \frac{d+1}{2} p)} \alpha^{-p} u_k(\alpha)^{1 - \frac{1}{p'}} \sum_{B \in \mathfrak{B}^k} \#(\mathcal{E} \cap B) \lesssim 2^{k(-1 + \frac{d+1}{2} p)} \alpha^{-p} \#\mathcal{E}
\]

since $u_k(\alpha) \geq 1$ and $R_B \geq 1$.

**Proof of (23).** We check from (19) that
\[
2^{kd} u_k(\alpha)^{2/\ell} \alpha^{-2} = 2^{k(-1 + \frac{d+1}{2} p)} \alpha^{-p}.\]

Thus by Tshebyshev’s inequality it suffices to prove
\[
(25) \quad \left\| \sum_{j \in \mathcal{E}_*} S_j \right\|_2^2 \lesssim 2^{kd} u_k(\alpha)^{2/\ell} \#\mathcal{E}_*.
\]
We set
\[
L := u_k(\alpha)^{2/\ell},
\]
\[
I(n, L) := \{n2^{-k}L, (n + 1)2^{-k}L\},
\]
\[
E(n, L) := \{\tilde{\epsilon} \in \mathcal{E} : \tilde{\epsilon}_{d+1} \in I(n, L)\},
\]
\[
\mathcal{S}_n := \sum_{\tilde{\epsilon} \in \mathcal{E}(n, L)} S_\tilde{\epsilon}.
\]

Now
\[
\left\| \sum_{\tilde{\epsilon} \in \mathcal{E}_*} S_\tilde{\epsilon} \right\|_2^2 \lesssim \sum_n \sum_{\tilde{n} : |n - \tilde{n}| \leq 4} (\mathcal{S}_n, \mathcal{S}_{\tilde{n}}) + \sum_n \sum_{\tilde{n} : |n - \tilde{n}| > 4} (\mathcal{S}_n, \mathcal{S}_{\tilde{n}}) =: I + II.
\]

For \( I \) we use the Schwarz inequality and then (17) to get
\[
|I| \lesssim \sum_n \|\mathcal{S}_n\|_2^2 = \sum_n \left\| \sum_{\tilde{\epsilon}_{d+1} \in I(n, L)} \sum_{\tilde{\epsilon}' : \tilde{\epsilon}'_{d+1} = \tilde{\epsilon}_{d+1}} S_{(\tilde{\epsilon}', \tilde{\epsilon}_{d+1})} \right\|_2^2
\]
\[
\lesssim L \sum_n \sum_{\tilde{\epsilon}_{d+1} \in I(n, L)} \left\| \sum_{\tilde{\epsilon}' : \tilde{\epsilon}'_{d+1} \in \mathcal{E}_*} S_{(\tilde{\epsilon}', \tilde{\epsilon}_{d+1})} \right\|_2^2
\]
\[
\lesssim L \sum_n \sum_{\tilde{\epsilon}_{d+1} \in I(n, L)} \sum_{\tilde{\epsilon}', \tilde{\epsilon} : |\tilde{\epsilon}'_{d+1}| \leq \tilde{\epsilon}_{d+1}} 2^{kd} (1 + 2^k|\tilde{\epsilon}' - \tilde{\epsilon}|) \lesssim L 2^{kd} \#\mathcal{E}.
\]

For \( II \) we use (18) and estimate
\[
|II| \lesssim \sum_{\tilde{\epsilon} \in \mathcal{E}_*: |\tilde{\epsilon}_{d+1} - \tilde{\epsilon}_{d+1}| \geq 2^{-k}L} \sum_{\tilde{\epsilon}' \in \mathcal{E}_*: |\tilde{\epsilon}'_{d+1} - \tilde{\epsilon}_{d+1}| \geq 2^{-k}L} |(S_{\tilde{\epsilon}}, S_{\tilde{\epsilon}'})| \lesssim \sum_{\tilde{\epsilon} \in \mathcal{E}_* : |\tilde{\epsilon}_{d+1} - \tilde{\epsilon}_{d+1}| \geq 2^{-k}L} \frac{2^{kd}}{(1 + 2^k|\tilde{\epsilon} - \tilde{\epsilon}'|)^{\ell/2}}.
\]

By (21) we have for \( R \geq 1 \) and fixed \( \tilde{\epsilon} \)
\[
\sum_{\tilde{\epsilon} \in \mathcal{E}_*: 2^{-k}R \leq |\tilde{\epsilon} - \tilde{\epsilon}'| \leq 2^{1-k}R} (1 + 2^k|\tilde{\epsilon} - \tilde{\epsilon}'|)^{-\ell/2} \lesssim u_k(\alpha) R^{1-\frac{k}{2}}
\]
and since \( \ell/2 > 1 \) we get (after setting \( R = 2^m \) and summing over \( m \) with \( 2^m \geq L \))
\[
|II| \lesssim 2^{kd} u_k(\alpha) L^{1-\frac{k}{2}} \#\mathcal{E}.
\]

Hence
\[
I + II \lesssim 2^{kd} \#\mathcal{E}(L + u_k(\alpha) L^{1-\frac{k}{2}})
\]
and with the optimal choice of \( L = [u_k(\alpha)]^{2/\ell} \) we obtain (25). \( \square \)

2.4. Proof that Proposition 2.2 implies (6). We shall first assume that in (2)
\[
(26) \quad \chi_k(z, y, 2^{-k} \xi) = \eta_k(z, 2^{-k} \xi) \chi(0, y)
\]
where \( \chi_0 \in C^\infty_c(\mathbb{R}^d) \) is supported on a small neighborhood of the origin but so that \( \chi_0(y) = 1 \) for \( y \in E \); moreover \( \eta_k \) is compactly supported in a set of diameter \( O(\varepsilon^2) \) near \((z, \xi) = (0, e_1)\), and the derivatives of \( \eta_k \) up to order 10d are uniformly bounded.
Let $Q_j = \prod_{i=1}^{d+1} [3^i, 3^i + 2^{-k}]$. For $m \geq 0$ let
\begin{align*}
\mathcal{E}_m &= \left\{ \xi \in \mathbb{Z}^d : 2^{-k(d+1)} - m - 1 < \left| Q_j \cap E \right| \leq 2^{-k(d+1)} - m \right\}, \\
E_m &= \bigcup_{j \in \mathcal{E}_m} Q_j \cap E.
\end{align*}
(27)
And we also set
\begin{align*}
a_{k,\mathcal{E}_m}(\xi) &= 2^{m+(k+1)d} \int_{Q_j \cap \mathcal{E}_m} \eta_k(z, 2^{-k}\xi) e^{i\varphi(z,\xi)} dz, \\
S_{\mathcal{E}_m}(y) &= \int a_{k,\mathcal{E}_m}(\xi) e^{i(y,\xi) - \varphi(z,\xi)} d\xi.
\end{align*}
Then it follows that
\begin{align*}
T_k^* \chi_E(y) &= \sum_{m=0}^{\infty} 2^{-m-(k+1)d} S_{\mathcal{E}_m}(y).
\end{align*}
Since $\partial_k^\alpha (\varphi(\xi,\xi) - \varphi(z,\xi)) = O(2^{-k})$ for any multiindex $\alpha$ it is easy to see that $a_{.,\mathcal{E}_m}$ satisfies (7) uniformly in $m$. Hence, the result of Proposition 2.2 can be applied to $\sum_{j \in \mathcal{E}_m} S_{\mathcal{E}_m}(y)$ and we get uniform bounds. Thus
\begin{align*}
\left\| \sum_{j \in \mathcal{E}_m} S_{\mathcal{E}_m} \right\|_{L^{p',\infty}} \lesssim 2^{k(\frac{d+1}{p'} - \frac{1}{p})} (\# \mathcal{E}_m)^{1/p} \lesssim 2^{k(\frac{d+1}{p'} - \frac{1}{p})} 2^m/p (2^{k(d+1)} \text{meas}(E_m))^{1/p}
\end{align*}
with implicit constants independent of $m$. For the second inequality we use
\begin{align*}
\# \mathcal{E}_m \lesssim 2^m 2^{k(d+1)} \text{meas}(E_m)
\end{align*}
which follows from (27). Consequently we get
\begin{align*}
\left\| T_k^* \chi_E \right\|_{L^{p',\infty}} \lesssim \sum_{m=0}^{\infty} 2^{-m-(k+1)d} \left\| S_{\mathcal{E}_m} \right\|_{L^{p',\infty}} \\
\lesssim \sum_{m=0}^{\infty} 2^{-m(1 - \frac{1}{p'})} 2^{k(\frac{d+1}{p'} - \frac{d+1}{2})} (\text{meas}(E_m))^{1/p} \lesssim 2^{k(\frac{d+1}{p'} - \frac{d+1}{2})} (\text{meas}(E))^{1/p}
\end{align*}
which is the desired estimate.

Finally we have to remove the assumption (26). Here one uses Fourier series in $y$ and expands $\chi_k(z, y, 2^{-k}\xi) = \sum_{\nu \in \mathbb{Z}^d} c_{k,\nu} \eta_k(z, 2^{-k}\xi) e^{i(y,\nu)}$ where the functions $\eta_k(z, 2^{-k}\xi)$ are as before but now with a bound that decays fast in $\nu$. We note that multiplication with $e^{i(y,\nu)}$ does not affect the $L^{p',1}$ norm, apply the previous bounds to the summands and sum in $\nu$ using the rapid decay in $\nu$. □

3. Combining the frequency localized pieces

We now combine the previous estimates on the operators $T_k$ and prove the following result in Triebel-Lizorkin spaces $F^q_{a,s}$. Recall ([19]) that if $\{L_k\}_{k=0}^\infty$ is a standard inhomogeneous dyadic frequency decomposition then the norm $\|f\|_{F^q_{a,s}}$ can be defined as the $L^q(\ell^s)$ norm of the sequence $\{2^q a_k L_k f\}$. In view of the embeddings $L^q = F^q_{0,2} \subset F^q_{0,q}$ for $q \geq 2$, $L^q \supset F^q_{0,r}$ for $r \leq 2$ the following result sharpens Theorem 1.1. For the case $r \geq 1$ one could argue by duality and follow [6] but we shall rely on a result in [15] which gives an estimate for all $r > 0$. 
Theorem 3.1. Let \( \ell \geq 3 \), let \( \mathcal{Z}, \mathfrak{y} \) be coordinate patches in \( \mathbb{R}^{d+1}, \mathbb{R}^d \), resp., let \( F \in \mathcal{F} \), with Schwartz kernel compactly supported in \( \mathcal{Z} \times \mathfrak{y} \), and let \( \mathcal{C} \) satisfy hypothesis \( \mathcal{H}(\ell) \). Suppose \( \frac{d}{c} < q < \infty \), \( \alpha = \mu + b + d(1/2 - 1/q) - 1/2 \) and \( r > 0 \). Then \( F \) maps \( F^q_{a,q}(\mathbb{R}^d) \) boundedly to \( F^a_{b,r}(\mathbb{R}^{d+1}) \).

We state (a slight variant of) the result from [15]. In this setting one is given operators \( T_k \) defined on the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \),

\[
T_k f(z) = \int K_k(z, y) f(y) \, dy, \quad z \in \mathbb{R}^d
\]

and each \( K_k \) is continuous and bounded. Let \( \zeta \in \mathcal{S}(\mathbb{R}^d) \) and \( \zeta_k(z) = 2^{kd} \zeta(2^k z) \), and define \( P_k g = \zeta_k * g \).

Theorem 3.2 ([15]). Let \( d_1 \leq d_2, 0 < \gamma < d_2, \varepsilon > 0, 1 < q_0 < q < \infty \), and assume

\[
\sup_{k > 0} 2^{k \gamma / q_0} \| T_k \|_{L^{q_0}(\mathbb{R}^{d_1}) \to L^{q_0}(\mathbb{R}^{d_2})} < \infty ,
\]

\[
\sup_{k > 0} \| T_k \|_{L^{\infty}(\mathbb{R}^{d_1}) \to L^{\infty}(\mathbb{R}^{d_2})} < \infty .
\]

Furthermore assume that for each cube \( Q \) there is a measurable set \( \mathcal{W}_Q \subset \mathbb{R}^d \) so that

\[
\text{meas}(\mathcal{W}_Q) \leq C \max\{|Q|^{1 - \gamma / d_2}, |Q|\}
\]

and there is \( \delta > 0 \) such that for every \( k \in \mathbb{N} \) and every cube \( Q \) with \( 2^k \text{diam}(Q) \geq 1 \)

\[
\sup_{x \in Q} \int_{\mathbb{R}^d \setminus \mathcal{W}_Q} |K_k(x, y)| \, dy \leq C \max\{(2^k \text{diam}(Q))^{-\delta}, 2^{-kd}\}.
\]

Then for \( q_0 < q < \infty \), \( r > 0 \)

\[
\left\| \left( \sum_k 2^{k \gamma / q} |P_k T_k f_k|^r \right)^{1/r} \right\|_q \lesssim \left( \sum_k \| f_k \|_q^q \right)^{1/q}.
\]

This (or a slightly sharper version) was formulated in [15] only for the case \( d_1 = d_2 \), but the result there implies the version cited above. Indeed if \( d_1 < d_2 \) we can define an operator \( \mathcal{T}_k \) on functions \( F \) on \( \mathbb{R}^{d_1} \times \mathbb{R}^{d_2 - d_1} \) by

\[
\mathcal{T}_k F(z) = \int K_k(z, y) \chi(w) F(y, w) \, dy \, dw
\]

where \( \chi \) is a nontrivial \( C^\infty(\mathbb{R}^{d_2 - d_1}) \) function. The assumptions on \( T_k \) imply the corresponding assumptions on \( \mathcal{T}_k \), by Minkowski’s and Hölder’s inequalities. Thus the equidimensional case in [15] may be applied and in the conclusion we specialize to tensor products, \( F(y, w) = f(y) \chi(w) \) we get the above generalization.

In order to prepare for our application of Theorem 3.2 we let \( T_k f(z) = \int K_k(z, y) f(y) \, dy \) with \( K_k \) as in (2). Let \( \beta_0 \) be a \( C^\infty \)-function supported in \( \{ \eta \in \mathbb{R}^{d+1} : |\eta| \leq 3/2 \} \) so that \( \beta_0(\eta) = 1 \) for \( |\eta| \leq 1 \); and, for \( k \geq 1 \) let \( \beta_k(\eta) = \beta_0(2^{-k}(\eta)) - \beta_0(2^{1-k}(\eta)) \). Define \( L_k \) on functions on \( \mathbb{R}^{d+1} \) by \( \mathcal{L}_k g = \beta_k(\eta) \tilde{g}(\eta) \). We use calculations in [8] and first observe that there is a constant \( A_0 > 1 \) so that

\[
\left\| \mathcal{L}_k T_k \right\|_{L^q \to L^0} \leq C_N \min\{2^{-kN}, 2^{-\tilde{k}N}\} \quad \text{if } |k - \tilde{k}| \geq A_0.
\]
This follows from the assumption that \( C \) does not meet the zero sections, which, by homogeneity implies that \( c_1|\xi| \leq |\phi'_\rho(z, \xi)| \leq C_1|\xi| \). The kernel \( K_{k\tilde{k}} \) of \( L_k T_k \) is given by
\[
K_{k\tilde{k}}(z, y) = \int \int \int \beta_k \eta \chi_k(z, y, 2^{-k} \xi) e^{i(z-w, \eta) + \phi(w, \xi) - (y, \xi)} dw d\eta d\xi
\]
and if \( k - \tilde{k} \) are sufficiently large then \( |\eta + \phi'_\rho(w, \xi)| \approx \max\{|\xi|, |\eta|\} \) on the support of the amplitude. Thus in this case we may use an integration by parts in \( w \) (followed by an integration by parts in \( \xi \) when \( z \) is large) to show that the kernels \( K_{k\tilde{k}} \) of \( L_k T_k \) satisfy the estimate
\[
|K_{k\tilde{k}}(z, y)| \leq C N 2^{-kN}(1 + |z|)^{-N}
\]
and vanish for \( y \) in the complement of a fixed compact set. Thus (32) follows. Similarly if \( \{\mathcal{L}_k\}_{k=0}^{\infty} \) is the corresponding frequency decomposition in \( \mathbb{R}^d \) we also see that \( T_k \mathcal{L}_{k\tilde{k}} \) has \( L^q(\mathbb{R}^d) \) to \( L^q(\mathbb{R}^{d+1}) \) operator norm \( \leq C N \min\{2^{-kN}, 2^{-\tilde{k}N}\} \) if \( |k - \tilde{k}| > 4 \).

From these preliminary remarks it follows quickly that for the proof of Theorem 3.1 it suffices to prove the inequalities
\[
\left\| \left( \sum_k 2^{k\rho} |L_{k+i_1} T_k \mathcal{L}_{k+i_2} g|^r \right)^{1/r} \right\|_q \lesssim \left( \sum_k 2^{k\rho} \|\mathcal{L}_{k+i_2} g\|^q \right)^{1/q}, \quad |i_1| \leq A_0, \quad |i_2| \leq 4
\]
with \( a = \mu + b + d(1/2 - 1/q) - 1/2 \). Setting \( T_k = 2^{-k\frac{d+1}{2}} T_k \) and \( f_k = 2^{k\rho} \mathcal{L}_{k+i_2} g \), the preceding inequality follows from
\[
\left( \sum_k 2^{k\rho} |L_{k+i_1} T_k f_k|^r \right)^{1/r} \lesssim \left( \sum_k \|f_k\|^q \right)^{1/q}, \quad |i_1| \leq A_0,
\]
for \( q_0 < q < \infty \), where \( q_0 > \frac{2\mu}{\gamma - 2} \). This in turn follows from an application of Theorem 3.2 with
\( d_1 = d, \quad d_2 = d + 1, \quad \gamma = \delta, \quad \delta = d \), and \( P_k = L_{k+i_1} \). The hypothesis (29) follows from the calculations in [16] (cf. also §2.1 above). The hypothesis (28) for \( \frac{2\mu}{\gamma - 2} < q < \infty \) follows from Theorem 2.1 and (29) by interpolation.

If \( \text{diam}(Q) > \varepsilon \) then \( W_Q \) is simply an expanded cube and (31) follows by the support assumption of \( K_k \). If \( \text{diam}(Q) < \varepsilon \) the exceptional sets are formed as in §2.1. For \( \theta \in S^{d-1} \) we set
\[
W_\theta(z_Q, C) = \{ y : |\phi(z_Q, \theta) - y, \theta| \leq C \text{diam}(Q), \quad |\Pi_{\theta^\perp}(\phi(z, \theta) - y) \leq C(\text{diam}(Q))^{1/2} \}
\]
and if \( \Theta_Q \) is a maximal set of \( (\text{diam}(Q))^{1/2} \) separated unit vectors we set
\[
W_Q = \bigcup_{\theta \in \Theta_Q} W_\theta(z_Q, C).
\]
Then the measure of \( W_Q \) is \( O(\text{diam}(Q)) = O(|Q|^{1-\frac{d}{d+1}}) \) so that (30) holds. The hypothesis (31) (even with large \( \delta \)) holds by the calculations in §2.1 (cf. (14)).

4. Remarks on the Constant Coefficient Case

We now let \( \rho \) be a \( C^\infty(\mathbb{R}^d \setminus \{0\}) \) function which is homogeneous of degree 1, so that \( \rho(\xi) \neq 0 \) for \( \xi \neq 0 \). We are interested in space time estimates for \( U_t \equiv e^{-it\rho(D)} \) defined by
\[
\widehat{U_t f}(\xi) = e^{it\rho(\eta)} \hat{f}(\xi)
\]
and obtain a result under a decay assumption for the Fourier transform of surface carried measure on
\[
\Sigma_\rho = \{ \xi : \rho(\xi) = 1 \}.
\]
Theorem 4.1. Let $\kappa > 1$ and let $\rho$ be as above such that the surface measure $d\sigma$ of $\Sigma_\rho$ satisfies
\begin{equation}
\sup_\xi (1 + |\xi|)^\kappa |\hat{d}\sigma(\xi)| < \infty.
\end{equation}

Let $\frac{2\kappa}{\kappa - 1} < q < \infty$ and let $I$ be a compact time interval. There is $C > 0$ such that
\begin{equation}
\left(\int_I \|e^{it\rho(D)} f\|_{L^q(\mathbb{R}^d)}^q dt\right)^{1/q} \leq C \|f\|_{B^0_{\alpha,q}(\mathbb{R}^d)}, \quad \alpha = \frac{d-1}{2} - \frac{d}{q},
\end{equation}
for all $f \in B^0_{\alpha,q}(\mathbb{R}^d)$.

Remark. Under the assumption that $\Sigma_\rho$ has nonvanishing curvature everywhere this follows from Theorem 3.1 above. We note that in this particular case a weaker result with $\alpha > (d-1)/2 - d/q$ is already in [7].

Sketch of proof. We will assume that $I = [-1, 1]$, as one can use rescaling to reduce to this case. Fix $k$ and define $S_{x,t} \equiv S_{x,t}^k$ by
\begin{equation}
S_{x,t}(y) = \int e^{i(x-y,\xi)+it\rho(\xi)} \chi(2^{-k}\xi) \, d\xi
\end{equation}
and as before we may assume that the support of $\chi$ has diameter $\leq \varepsilon^2$, for sufficiently small $\varepsilon > 0$, and is contained in $\{1/2 < |\xi| \leq 2\}$. Fix $\xi_0 \in \Sigma_\rho$ and $\rho_0 > 0$ so that $\rho_0\xi_0 \in \text{supp}(\chi)$. Let $u \mapsto \Xi(u)$ be a parametrization of $\Sigma_\rho$ near $\xi_0$ with parameter $u \in \mathbb{R}^{d-1}$ near $u_0$, and $\Xi(u_0) = \xi_0$. Let $n_0$ be the outer normal unit vector to $\Sigma_\rho$ at $\xi_0$. Let $\Gamma$ be the cone formed by the $(\rho\Xi(u), \rho)$ with $\rho > 0$ and $u$ near $u_0$. Let $\tilde{N}_0 = n_0 - \langle \xi_0, n_0 \rangle \tilde{e}_{d+1}$, which is a normal vector to $\Gamma$ at $(\rho\Xi(u_0), \rho)$. By finite decompositions of $\chi$ we may further assume that
\begin{equation}
S_{x,t}(y) = \int \beta(2^{-k}\xi)e^{i(x-y,\xi)+it\rho(\xi)} \, d\xi
\end{equation}
where $\beta \in C^\infty_c$ supported in an $\varepsilon^2$-neighborhood of $\rho_0\Xi(u_0)$.

The proof of Theorem 4.1 is a straightforward variant of the proof of Theorem 3.1 once we have established the two appropriate replacements for the scalar product bounds (18) and (17), namely
\begin{equation}
|\langle S_{x,t}, S_{x,t} \rangle| \lesssim 2^{kd}(1 + 2^k|x - \tilde{x}| + 2^k|t - \tilde{t}|)^{-\kappa},
\end{equation}
and a better estimate when $(x - \tilde{x}, t - \tilde{t})$ is orthogonal (or near orthogonal) to $\tilde{N}_0$:
\begin{equation}
|\langle S_{x,t}, S_{x,t} \rangle| \leq C_M 2^{kd}(1 + 2^k|x - \tilde{x}| + 2^k|t - \tilde{t}|)^{-M}
\end{equation}
if $|(x - \tilde{x}, n_0) - (t - \tilde{t})\langle \xi_0, n_0 \rangle| \leq \varepsilon_o |(x - \tilde{x}, t - \tilde{t})|
for a small $\varepsilon_o > 0$.

As before $2\pi^d \langle S_{x,t}, S_{x,t} \rangle = \langle \hat{S}_{x,t}, \hat{S}_{x,t} \rangle$. We scale and then use generalized polar coordinates $\xi = \rho\Xi(u)$ to write
\begin{equation}
\langle \hat{S}_{x,t}, \hat{S}_{x,t} \rangle = 2^{kd} \iint b(\rho, u) e^{i2^k\rho(x - \tilde{x}, \Xi(u)) + t\tilde{t})} du \, d\rho
\end{equation}
where $b$ is smooth and supported near $(\rho_0, u_0)$. Now for any $\chi \in C^\infty_c$ we have $|\chi d\sigma(\xi)| \lesssim (1 + |\xi|)^{-\kappa}$, by assumption (34). We apply this to the inner integral of (37) and obtain
\begin{equation}
|\langle S_{x,t}, S_{x,t} \rangle| \lesssim 2^{kd}(1 + 2^k|x - \tilde{x}|)^{-\kappa}.
\end{equation}
Let \( B = 2 \max\{|\xi| : \xi \in \Sigma_0\} \). By an integration by parts in \( \rho \) (after interchanging the order of integration in (37)) we obtain
\[
|\langle S_{x,t}, S_{\tilde{x},\tilde{t}} \rangle| \leq C_N 2^{kd}(1 + 2^k|t - \tilde{t}|)^{-N} \quad \text{if } |t - \tilde{t}| \geq B|x - \tilde{x}|,
\]
and (35) follows from (38) and (39).

We now prove (36) and in view of (39) we may assume \(|t - \tilde{t}| < B|x - \tilde{x}|\). We distinguish two cases. In the first case we assume \(|\langle \frac{x - \tilde{x}}{|x - \tilde{x}|}, \mathbf{n}_0 \rangle| \leq 1 - \varepsilon\). We note that \(|\Xi(u) - \Xi(u_0)| = O(\varepsilon^2)\) and \(|(\nabla_u \langle x - \tilde{x}, \Xi(u) \rangle)_{u = u_0}| \sim |\Pi_{n_0^+}^\perp (x - \tilde{x})|\). Hence we have \(|\nabla_u \langle x - \tilde{x}, \Xi(u) \rangle| \geq \varepsilon|x - \tilde{x}|\) on the support of \( b \) provided that \( \varepsilon \) is sufficiently small, and higher derivatives of \( \langle x - \tilde{x}, \Xi(u) \rangle \) are \( O(|x - \tilde{x}|) \). Thus, integrating by parts in the inner \( u \)-integral in (37), we get (36). We now consider the second case \(|\langle \frac{x - \tilde{x}}{|x - \tilde{x}|}, \mathbf{n}_0 \rangle| \geq 1 - \varepsilon\). This means that \( \frac{x - \tilde{x}}{|x - \tilde{x}|} = s \mathbf{n}_0 + O(\varepsilon) \) where \( s = 1 \) or \( s = -1 \). From the condition on \( (x - \tilde{x}, t - \tilde{t}) \) in (36) we see that the \( \rho \)-derivative of the phase is
\[
\langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t} = \langle x - \tilde{x}, \xi_0 \rangle + \frac{\langle x - \tilde{x}, \mathbf{n}_0 \rangle}{\langle \xi_0, \mathbf{n}_0 \rangle} + O((\varepsilon^2 + \varepsilon_0)|x - \tilde{x}|)
\]
\[
= s|x - \tilde{x}|\left(\langle \xi_0, \mathbf{n}_0 \rangle + \frac{1}{\langle \xi_0, \mathbf{n}_0 \rangle} \right) + O((\varepsilon + \varepsilon_0)|x - \tilde{x}|).
\]
Now \(|\langle \xi_0, \mathbf{n}_0 \rangle| \geq c > 0\) which is a consequence of the homogeneity relation \( \rho(\xi) = \langle \xi, \nabla \rho(\xi) \rangle \). Hence, \(|\langle x - \tilde{x}, \Xi(u) \rangle + t - \tilde{t}| \gtrsim |x - \tilde{x}|\) if \( \varepsilon \) and \( \varepsilon_0 \) are small enough, and another integration by parts in \( \rho \) gives (36).

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